1 Trigonometry

What You'll Learn

To determine the measures of sides and angles in right, acute, and obtuse triangles and to solve related problems

And Why

Applications of trigonometry arise in land surveying, navigation, cartography, computer graphics, machining, medical imaging, and meteorology, where problems call for calculations involving angles, lengths, and distances using indirect measurements.

Key Words

- primary trigonometric ratios
- sine
- cosine
- tangent
- angle of inclination
- angle of depression
- acute triangle
- obtuse triangle
- oblique triangle
- Sine Law
- Cosine Law

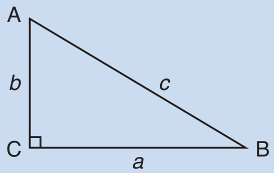
CHAPTER 1 Activate Prior Knowledge

The Pythagorean Theorem

Prior Knowledge for 1.1

The **hypotenuse** of a right triangle is the side opposite the right angle. It is the longest side.

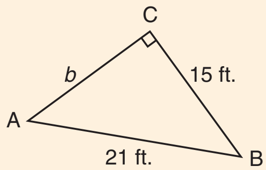
Pythagorean Theorem
In right △ABC with hypotenuse c:
$c^2 = a^2 + b^2$



Example

Materials
- scientific calculator

Determine the unknown length b.



Solution

Use the Pythagorean Theorem in △ABC.

$c^2 = a^2 + b^2$	Substitute: $c = 21$ and $a = 15$
$21^2 = 15^2 + b^2$	Subtract 15^2 from both sides to isolate b^2.
$21^2 - 15^2 = b^2$	Take the square root of both sides to isolate b.
$b = \sqrt{21^2 - 15^2}$	Press: 2nd x^2 21 x^2 − 15 x^2) ENTER
$\doteq 14.70$	

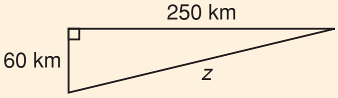
So, side b is about 15 feet long.

Write your answer to the same number of decimal places as the least accurate measurement used in calculations.

CHECK ✓

1. Determine each unknown length.

a)



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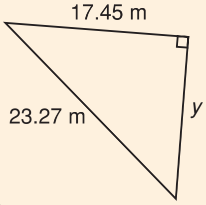
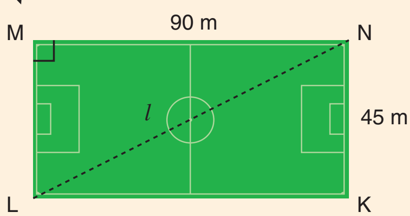


2. In isosceles right △PQR, ∠P = 90°, PR = 6.7 m. Determine the length of QR.

3. What is the length of the diagonal l across the soccer field?



Metric and Imperial Unit Conversions

The metric system is based on powers of 10.

Metric conversions

1 cm = 10 mm	1 m = 100 cm	1 km = 1000 m
1 mm = 0.1 cm	1 cm = 0.01 m	1 m = 0.001 km

The most common imperial units of length are the inch, foot, yard, and mile.

Imperial conversions

1 foot = 12 inches	1 yard = 3 feet	1 mile = 5280 feet
	1 yard = 36 inches	1 mile = 1760 yards

Example

Write each pair of measures using the same unit.

a) 54 cm, 3.8 m **b)** 22 inches, 12 feet 4 inches

Solution

a) 1 m = 100 cm; so, 3.8 m = 3.8×100 = 380 cm
Alternatively, 1 cm = 0.01 m, so 54 cm = 54×0.01 = 0.54 m

b) 1 foot = 12 inches, so 12 feet = 144 inches;
12 feet 4 inches = 144 inches + 4 inches = 148 inches
Alternatively, 22 inches = 1 foot 10 inches

CHECK ✓

1. Convert each metric measure to the unit indicated.

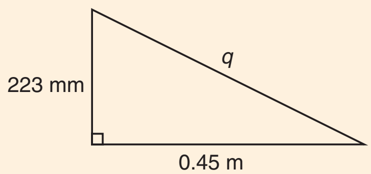
a) 7.2 cm to millimetres **b)** 9215 m to kilometres **c)** 9.35 km to metres
d) 832 cm to metres **e)** 879 m to centimetres **f)** 65 mm to metres

2. Convert each imperial measure to the unit indicated.

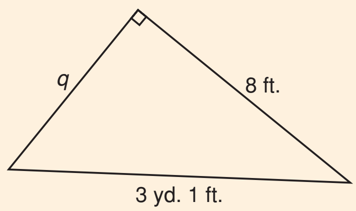
a) 7 feet to inches **b)** 28 yards to feet **c)** 8 miles to feet
d) 963 feet to yards **e)** 23 feet 5 inches to inches **f)** 48 inches to feet

3. Determine q. If you need to convert measurements to a different unit, explain why.

a)



b)



Home Quit

1.1 Trigonometric Ratios in Right Triangles

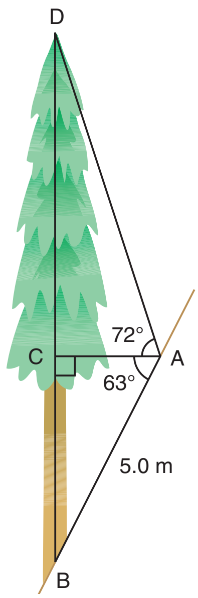
Specialists in forestry and arboriculture apply trigonometry to determine heights of trees. They may use a clinometer, an instrument for measuring angles of elevation.

Investigate

Choosing Trigonometric Ratios

Materials

- scientific calculator



Work with a partner.

An arborist uses a clinometer to determine the height of a tree during a hazard evaluation. This diagram shows the arborist's measurements.

- Use △ABC.
  Determine the lengths of BC and AC.
- Use △ACD.
  Determine the length of CD.
- What is the height of the tree?

For accuracy, keep more decimal places in your calculations than you need in the final answer.

Reflect

- Describe the strategies you used to determine the height of the tree. What angles and trigonometric ratios did you use?
- Compare your results and strategies with another pair. How are they similar? How are they different?

Connect the Ideas

Primary trigonometric ratios

The word "trigonometry" means "measurement of a triangle."

- The **primary trigonometric ratios** are sine, cosine, and tangent.

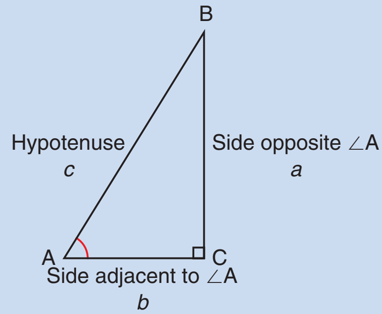
The primary trigonometric ratios

For acute ∠A in right △ABC:

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{a}{b}$$



Each vertex is labelled with a capital letter. Each side is labelled with the lowercase letter of the opposite vertex.

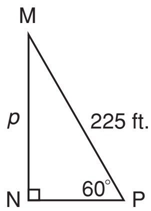
- We can use the primary trigonometric ratios or combinations of these ratios to determine unknown measures.

Example 1 Determining Side Lengths

Materials

- scientific calculator

Determine the length of p in △MNP.



Set your calculator to degree mode before using sine, cosine, or tangent.

Solution

In △MNP:

- The length of the hypotenuse is given.
- The measure of acute ∠P is given.
- p is opposite ∠P.

So, use the sine ratio.

$$\sin P = \frac{MN}{MP}$$ Substitute: MN = p, ∠P = 60°, and MP = 225

$$\sin 60^\circ = \frac{p}{225}$$ Multiply both sides by 225 to isolate p.

$$\sin 60^\circ \times 225 = p$$

Press: SIN 60) × 225 ENTER

$$p \doteq 194.86$$

The length of p is about 195 feet.

Write the length of p to the nearest foot because the length of n is to the nearest foot.

Inverse ratios

It the key strokes shown here do not work on your calculator, refer to the user manual.

- You can use the **inverse ratios** $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ to determine the measure of an angle when its trigonometric ratio is known. Press 2nd SIN, 2nd COS, or 2nd TAN to access the inverse ratios on a scientific calculator.

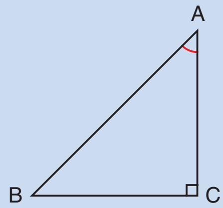
Inverse ratios

For acute $\angle A$ in right $\triangle ABC$:

$\sin^{-1}(\sin A) = A$

$\cos^{-1}(\cos A) = A$

$\tan^{-1}(\tan A) = A$



- In navigation and land surveying, direction is described using a **bearing**. The bearing is given as a three-digit angle between 000° and 360° measured clockwise from the north line.

Example 2 Determining Angle Measures

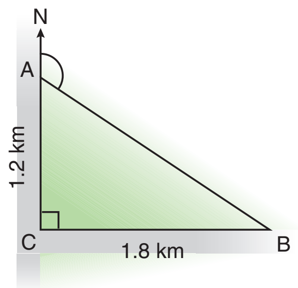
Materials

- scientific calculator

Michelle is drawing a map of a triangular plot of land.

a) Determine the angles in the triangle.

b) Determine the bearing of the third side of the plot, AB.



Solution

a) $\tan A = \dfrac{BC}{AC}$

$= \dfrac{1.8}{1.2}$

Substitute: BC = 1.8 and AC = 1.2

$\angle A = \tan^{-1}\left(\dfrac{1.8}{1.2}\right)$

$\doteq 56.31°$

Press: 2nd TAN 1.8 ÷ 1.2) ENTER

The measure of $\angle A$ is about 56°.

$\angle B = 180° - \angle C - \angle A$

$= 180° - 90° - 56°$

$= 34°$

The sum of the angles in a triangle is 180°.

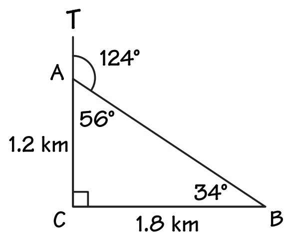
The measure of $\angle B$ is about 34°.

b) $\angle BAC$ and $\angle TAB$ form a straight line. So:

$\angle TAB = 180° - 56°$

$= 124°$

The bearing of the third side of the plot is about 124°.



Example 3

Materials

- scientific calculator

An angle of elevation is also called an *angle of inclination*.

Solving Problems

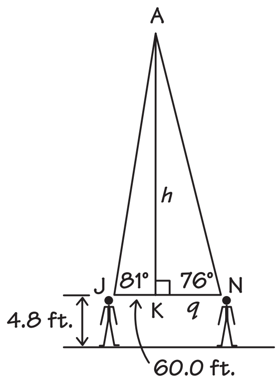
Jenny and Nathan want to determine the height of the Pickering wind turbine. Jenny stands 60.0 feet from the base of the turbine. She measures the angle of elevation to the top of the turbine to be 81°. On the other side of the turbine, Nathan measures an angle of elevation of 76°. Jenny and Nathan each hold their clinometers about 4.8 feet above the ground when measuring the angle of elevation.

a) What is the height of the wind turbine?

b) How far away from the base is Nathan?

Solution

Sketch and label a diagram. Drawing a diagram may help you visualize the information in the problem. Your diagram does not need to be drawn to scale. For example, this diagram is not drawn to scale.



a) Use $\triangle$AKJ to find the length of AK.

$$\tan J = \frac{AK}{JK}$$ Substitute: AK = h, $\angle$J = 81°, and JK = 60.0

$$\tan 81° = \frac{h}{60.0}$$ Multiply both sides by 60.0 to isolate h.

$$h = \tan 81° \times 60.0$$

Press: TAN 81) × 60.0 ENTER

$$h \doteq 378.83$$

The turbine is about 378.8 feet above eye level.

378.8 feet + 4.8 feet = 383.6 feet

The height of the wind turbine is about 384 feet.

This solution assumes that Jenny and Nathan are standing on level ground. Can you explain why?

b) In right $\triangle$AKN, the measure of $\angle$N and the length of the opposite side AK are known.

$$\tan N = \frac{AK}{KN}$$ Substitute: KN = q, $\angle$N = 76°, and AK $\doteq$ 378.8

$$\tan 76° = \frac{378.8}{q}$$ Rearrange the equation to isolate q.

$$q = \frac{378.8}{\tan 76°}$$

Press: 378.8 ÷ TAN 76) ENTER

$$q \doteq 94.4$$

Nathan is about 94 feet away from the base of the turbine.

Practice

A

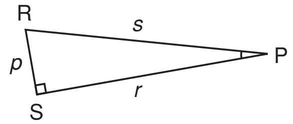
1. For each triangle, name each side in two different ways.

a) Hypotenuse

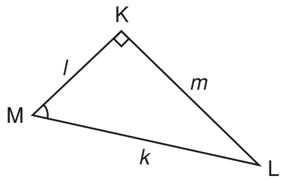
b) Side opposite the marked angle

c) Side adjacent to the marked angle

i)

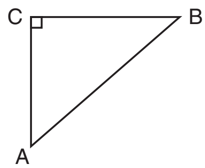


ii)



2. Write each trigonometric ratio as a ratio of sides.

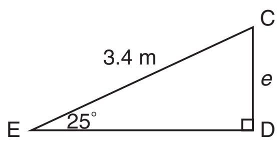
a) sin A **b)** cos A **c)** cos B **d)** tan B



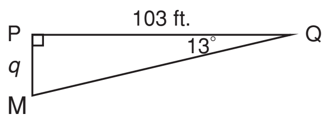
■ For help with questions 3 and 4, see Example 1.

3. Which primary trigonometric ratio can you use to calculate the length of each indicated side?

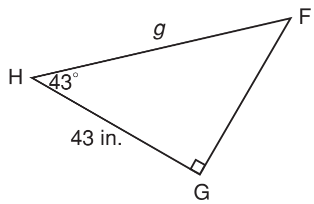
a)



b)



c)



Give answers to the same number of decimal places as the least accurate measurement used in calculations.

4. Use the ratios you found in question 3 to calculate the length of each indicated side.

■ For help with question 5, see Example 2.

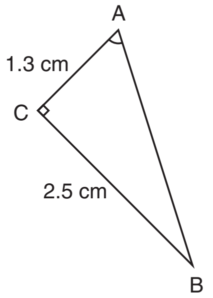
Give each angle measure to the nearest degree.

To *solve a triangle* means to determine the measures of all its sides and angles.

Use the *Course Study Guide* at the end of the book to recall any measurement conversions.

5. For each triangle, determine tan A. Then, determine the measure of ∠A.

a)



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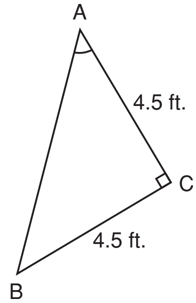
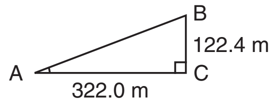
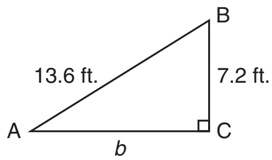


c)



6. Use trigonometric ratios and the Pythagorean Theorem to solve each triangle.

a)



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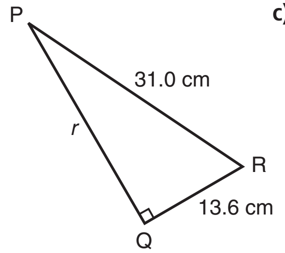

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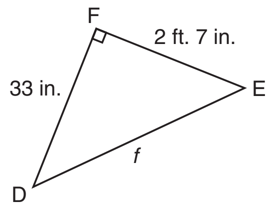
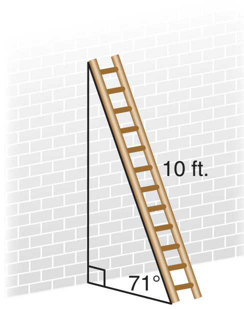


B

7. A ladder 10 feet long is leaning against a wall at a 71° angle.

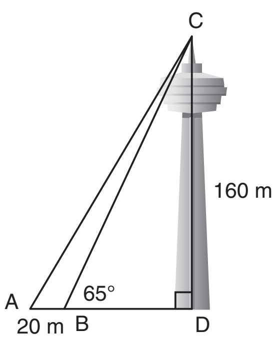
a) How far from the wall is the foot of the ladder?

b) How high up the wall does the ladder reach?



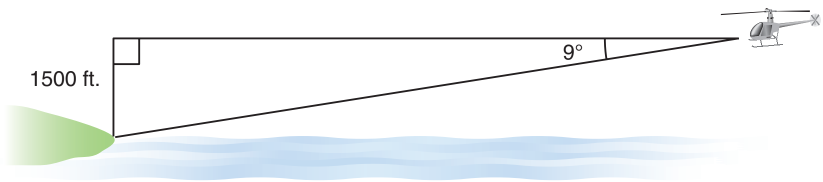
■ For help with questions 8 and 9, see Example 3.

8. The Skylon Tower in Niagara Falls is about 160 m high. From a certain distance, Frankie measures the angle of elevation to the top of the tower to be 65°. Then he walks another 20 m away from the tower in the same direction and measures the angle of elevation again. Use primary trigonometric ratios to determine the measure of the new angle of elevation.

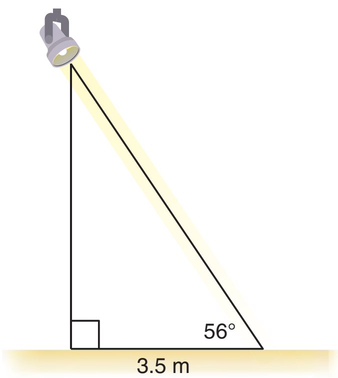


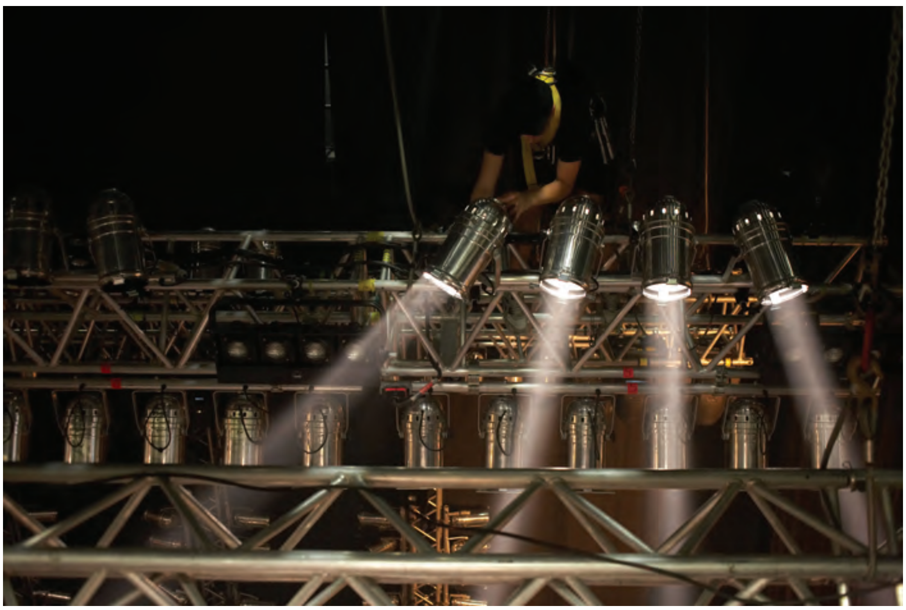
For help with question 9, see Example 3.

9. A rescue helicopter is flying horizontally at an altitude of 1500 feet over Georgian Bay toward Beausoleil Island. The angle of depression to the island is 9°. How much farther must the helicopter fly before it is above the island? Give your answer to the nearest mile.



10. A theatre lighting technician adjusts the light to fall on the stage 3.5 m away from a point directly below the lighting fixture. The technician measures the angle of elevation from the lighted point on the stage to the fixture to be 56°. What is the height of the lighting fixture?

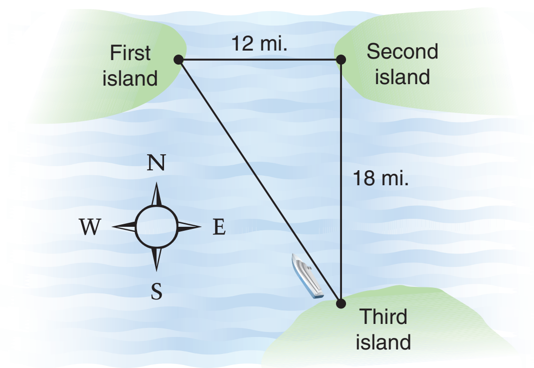




Assume Kenya measures the angle of elevation from the ground to the top of the tower.

11. Kenya's class is having a contest to find the tallest building in Ottawa. Kenya chose the Place de Ville tower. Standing 28.5 m from the base of the tower, she measured an angle of elevation of 72° to its top. Use Kenya's measurements to determine the height of the tower.

12. A ship's chief navigator is plotting the course for a tour of three islands. The first island is 12 miles due west of the second island. The third island is 18 miles due south of the second island.



a) Do you have enough information to determine the bearing required to sail directly back from the third island to the first island? Explain.

b) If your answer to part a is yes, describe how the navigator would determine the bearing.

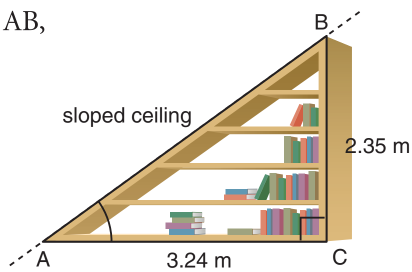
An *angle of inclination* measures an angle above the horizontal. An *angle of depression* measures an angle below the horizontal.

13. **Assessment Focus** A carpenter is building a bookshelf against the sloped ceiling of an attic.

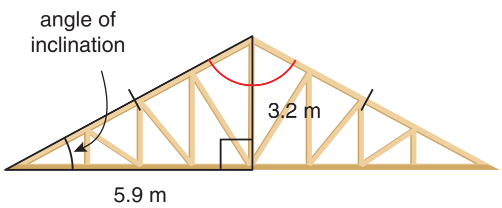
a) Determine the length of the sloped ceiling, AB, used to build the bookshelf.

b) Determine the measure of ∠A. Is ∠A an angle of inclination or an angle of depression? Why?

c) Describe another method to solve part b. Which method do you prefer? Why?



14. A roof has the shape of an isosceles triangle.



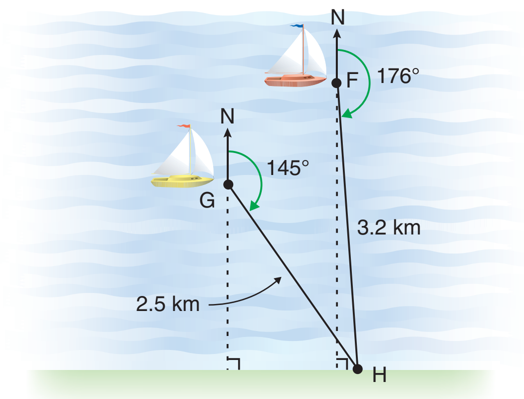
a) What is the measure of the angle of inclination of the roof?

b) What is the measure of the angle marked in red?

c) Write your own problem about the roof.
Make sure you can use the primary trigonometric ratios to solve it.
Solve your problem.

15. **Literacy in Math** In right $\triangle ABC$ with $\angle C = 90°$, $\sin A = \cos B$.
Explain why.

16. Two boats, F and G, sail to the harbour, H. Boat F sails 3.2 km on a bearing of 176°. Boat G sails 2.5 km on a bearing of 145°. Determine the distance from each boat straight to the shore.



C

17. Use paper and a ruler. Draw a right $\triangle ABC$ where:

a) $\sin A = \cos B = 0.5$

b) $\tan A = \tan B = 1.0$

Describe your method.

In Your Own Words

Explain why someone might need to use primary trigonometric ratios in daily life or a future career.

1.2 Investigating the Sine, Cosine, and Tangent of Obtuse Angles

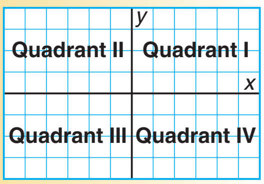
To create a proper joint between two pieces of wood, a carpenter needs to measure the angle between them. When corners meet at a right angle, the process is relatively simple. When pieces of wood are joined at an acute or an obtuse angle, the task of creating a proper joint is more difficult.

Inquire

Materials

- *The Geometer's Sketchpad* or grid paper and protractor
- TechSinCosTan.gsp
- scientific calculator

The intersection of the *x*-axis and *y*-axis creates four regions, called **quadrants**, numbered counterclockwise starting from the upper right.



The angle made by line segment *r* with the positive *x*-axis is labelled ∠A.

Exploring Trigonometric Ratios

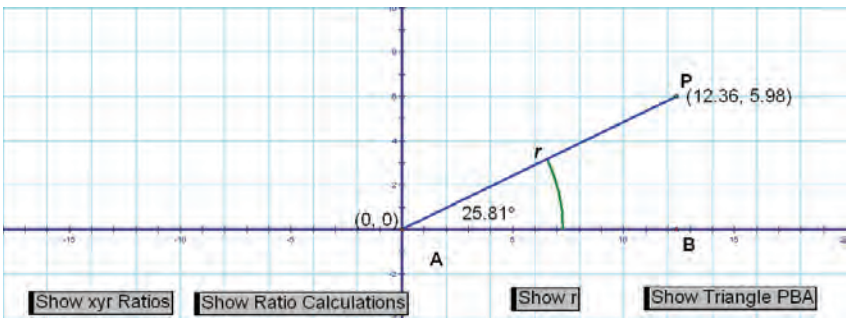
Choose *Using The Geometer's Sketchpad* or *Using Pencil and Paper*.

Using *The Geometer's Sketchpad*

Work with a partner.

Part A: Investigating Trigonometric Ratios Using Point P(*x*, *y*)

- Open the file *TechSinCosTan.gsp*.
  Make sure your screen looks like this.



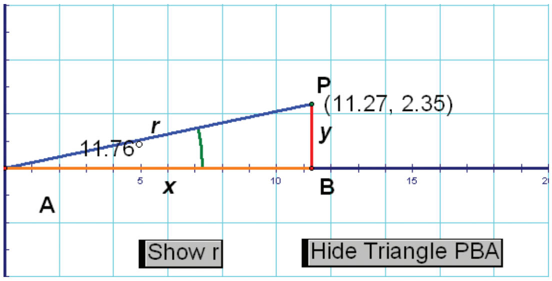
To deselect, use the **Selection Arrow**

and click anywhere on the screen.

1. Use the **Selection Arrow** tool. 

Click on **Show Triangle PBA**, then deselect the triangle.

2. Move point P(x, y) around in Quadrant I.



In right △PBA, how can you use the Pythagorean Theorem and the values of x and y to determine the length of side r?
Click on **Show r**. Compare your method with the formula on the screen.

3. Choose a position for point P in Quadrant I.
In right △PBA:

a) What is the measure of ∠A?

b) Which side is opposite ∠A? Adjacent to ∠A? Which side is the hypotenuse?

c) Use x, y, and r. Write each ratio.

i) sin A **ii)** cos A **iii)** tan A

Click on **Show xyr Ratios**.
Compare your answers with the ratios on the screen.

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}$$

$$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}}$$

$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}$$

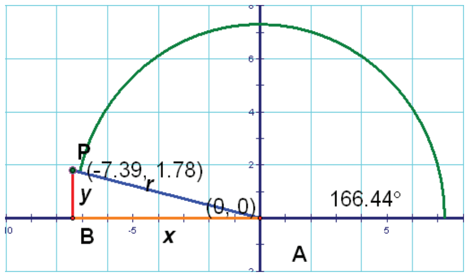
4. Choose a position for point P in Quadrant II.

a) What is the measure of ∠A?

b) In right △PBA, what is the measure of ∠PAB?

c) Is the *x*-coordinate of P positive or negative?

d) Is the *y*-coordinate of P positive or negative?



5. Copy the table. Use a scientific calculator for parts a and b.

	Angle measure	sin A	cos A	tan A
Acute ∠A				
Obtuse ∠A				

An acute angle is less than 90°. An obtuse angle is between 90° and 180°.

a) Use the measure of ∠A from question 3. Complete the row for acute ∠A.

b) Use the measure of ∠A from question 4. Complete the row for obtuse ∠A.

c) Click on **Show Ratio Calculations**. Compare with the results in the table.

Part B: Determining Signs of Trigonometric Ratios

- Use the **Selection Arrow** tool.
  Move point P around Quadrants I and II.

6. a) Which type of angle is ∠A if point P is in Quadrant I?

b) Which type of angle is ∠A if point P is in Quadrant II?

7. a) Can $r = \sqrt{x^2 + y^2}$ be negative? Why or why not?

b) When ∠A is acute, is *x* positive or negative?

c) When ∠A is obtuse, is *x* positive or negative?

d) When ∠A is acute, is *y* positive or negative?

e) When ∠A is obtuse, is *y* positive or negative?

8. Suppose $\angle A$ is acute. Use your answers from question 7.

a) Think about $\sin A = \frac{y}{r}$.
Is sin A positive or negative? Explain why.

b) Think about $\cos A = \frac{x}{r}$.
Is cos A positive or negative? Explain why.

c) Think about $\tan A = \frac{y}{x}$.
Is tan A positive or negative? Explain why.

d) Move point P around in Quadrant I. What do the sine, cosine, and tangent calculations show about your work for parts a, b, and c?

9. Suppose $\angle A$ is obtuse. Use your answers from question 7.

a) Think about $\sin A = \frac{y}{r}$.
Is sin A positive or negative? Explain why.

b) Think about $\cos A = \frac{x}{r}$.
Is cos A positive or negative? Explain why.

c) Think about $\tan A = \frac{y}{x}$.
Is tan A positive or negative? Explain why.

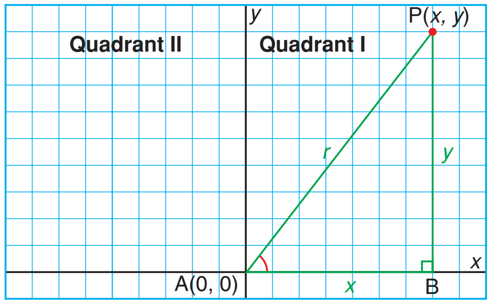
d) Move point P around in Quadrant II. What do the sine, cosine, and tangent calculations on screen show about your work for parts a, b, and c?

Using Pencil and Paper

Work with a partner.

Part A: Investigating Trigonometric Ratios Using Point P(x, y)

1. On grid paper, draw a point P(x, y) in Quadrant I of a coordinate grid. Label the sides and vertices as shown.



2. In right △PBA, how can you use the Pythagorean Theorem and the values of x and y to determine the number of units for side r?

$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}$

$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}}$

$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A}$

3. In right △PBA:

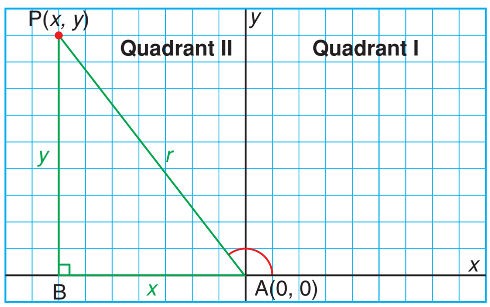
a) What is the measure of ∠A?

b) Which side is opposite ∠A? Adjacent to ∠A? Which side is the hypotenuse?

c) Use x, y, and r. Write each ratio.

i) sin A **ii)** cos A **iii)** tan A

4. On the same grid, choose a second position for point P in Quadrant II. Label as shown.



a) What is the measure of ∠A?

b) In right △PBA, what is the measure of ∠PAB?

c) Is the x-coordinate of P positive or negative?

d) Is the y-coordinate of P positive or negative?

5. Copy the table.

	Angle measure	sin A	cos A	tan A
Acute $\angle$A				
Obtuse $\angle$A				

Use a scientific calculator.

a) Use the measure of $\angle$A and the number of units for each side from question 3. Complete the row for acute $\angle$A.

b) Use the measure of $\angle$A and the number of units for each side from question 4. Complete the row for obtuse $\angle$A.

Part B: Determining Signs of the Trigonometric Ratios

An acute angle is less than 90°. An obtuse angle is between 90° and 180°.

6. a) Which type of angle is $\angle$A if point P is in Quadrant I?

b) Which type of angle is $\angle$A if point P is in Quadrant II?

7. a) Can $r = \sqrt{x^2 + y^2}$ be negative? Why or why not?

b) When $\angle$A is acute, is x positive or negative?

c) When $\angle$A is obtuse, is x positive or negative?

d) When $\angle$A is acute, is y positive or negative?

e) When $\angle$A is obtuse, is y positive or negative?

8. Suppose $\angle$A is acute. Use your answers from question 7.

a) Think about $\sin A = \frac{y}{r}$.
Is sin A positive or negative? Explain why.

b) Think about $\cos A = \frac{x}{r}$.
Is cos A positive or negative? Explain why.

c) Think about $\tan A = \frac{y}{x}$.
Is tan A positive or negative? Explain why.

9. Suppose $\angle$A is obtuse. Use your answers from question 7.

a) Think about $\sin A = \frac{y}{r}$.
Is sin A positive or negative? Explain why.

b) Think about $\cos A = \frac{x}{r}$.
Is cos A positive or negative? Explain why.

c) Think about $\tan A = \frac{y}{x}$.
Is tan A positive or negative? Explain why.

Practice

A

1. Use your work from *Inquire*. For each angle measure, is point P in Quadrant I or Quadrant II?

a) $35°$

b) $127°$

c) $95°$

2. Is the sine of each angle positive or negative?

a) $45°$

b) $67°$

c) $153°$

3. Is the cosine of each angle positive or negative?

a) $168°$

b) $32°$

c) $114°$

4. Is the tangent of each angle positive or negative?

a) $123°$

b) $22°$

c) $102°$

B

5. Is each ratio positive or negative? Justify your answers.

a) $\tan 40°$ b) $\cos 120°$

c) $\tan 150°$ d) $\sin 101°$

e) $\cos 98°$ f) $\sin 13°$

6. Is $\angle A$ acute, obtuse, or either? Justify your answers.

a) $\cos A = 0.35$

b) $\tan A = -0.72$

c) $\sin A = 0.99$

Reflect

- What did your results show about the measure of $\angle A$ and its trigonometric ratios when point P is in Quadrant I?
- What did your results show about the measure of $\angle A$ and its trigonometric ratios when point P is in Quadrant II?

1.3 Sine, Cosine, and Tangent of Obtuse Angles

Surveyors and navigators often work with angles greater than 90°. They need to know how to interpret their calculations.

Investigate

Exploring Supplementary Angles

Materials
- scientific calculator

The sum of the measures of two supplementary angles is 180°.

Work with a partner.

- Copy and complete the table for $\angle A = 25°$.

	$\angle A = 25°$	Supplementary angle
sin A		
cos A		
tan A		

- Repeat for $\angle B = 105°$ and for $\angle C = 150°$.
- Examine your tables for $\angle A$, $\angle B$, and $\angle C$. What relationships do you see between
  - The sines of supplementary angles?
  - The cosines of supplementary angles?
  - The tangents of supplementary angles?

Reflect

- If you know the trigonometric ratios of an acute angle, how can you determine the ratios of its supplementary angle?
- If you know the trigonometric ratios of an obtuse angle, how can you determine the ratios of its supplementary angle?

Connect the Ideas

xyr definition

The trigonometric ratios can be defined using a point P(x, y) on a coordinate grid.

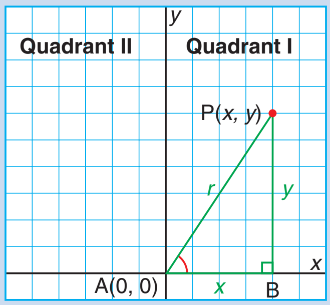
Trigonometric ratios

In right △PBA:

$\sin A = \frac{y}{r}$ $(r \neq 0)$

$\cos A = \frac{x}{r}$ $(r \neq 0)$

$\tan A = \frac{y}{x}$ $(x \neq 0)$



When ∠A is obtuse, x is negative.

Signs of the primary trigonometric ratios

When point P is in Quadrant I, ∠A is acute.

- sin A is positive.
- cos A is positive.
- tan A is positive.

When point P is in Quadrant II, ∠A is obtuse.

- sin A is positive.
- cos A is negative.
- tan A is negative.

Example 1 Determining Trigonometric Ratios of an Obtuse Angle

Materials

- scientific calculator

Suppose ∠C = 123°.

Determine each trigonometric ratio for ∠C, to 4 decimal places.

a) sin C **b)** cos C **c)** tan C

Solution

Use a calculator.

a) $\sin C = \sin 123° \doteq 0.8387$

b) $\cos C = \cos 123° \doteq -0.5446$

c) $\tan C = \tan 123° \doteq -1.5399$

Supplementary angles

The sum of the measures of two supplementary angles is 180°.

In Lesson 1.2, we investigated relationships between trigonometric ratios of an acute angle and its supplement. We can use these relationships to determine the measure of an obtuse angle.

Properties of supplementary angles

Given an acute angle, A, and its supplementary obtuse angle $(180° - A)$:

- $\sin A = \sin (180° - A)$
- $\cos A = -\cos (180° - A)$
- $\tan A = -\tan (180° - A)$

Example 2 Determining the Measure of an Obtuse Angle

Materials

- scientific calculator

Write the measure of each supplementary obtuse angle when:

a) The sine of acute $\angle P$ is 0.65.

b) The cosine of acute $\angle R$ is 0.22.

c) The tangent of acute $\angle S$ is 0.44.

Solution

Method 1

First determine the measure of the acute angle.

a) $\sin P = 0.65$

$$\angle P = \sin^{-1}(0.65)$$
$$\angle P \doteq 40.5°$$
$$180° - \angle P = 180° - 40.5°$$
$$= 139.5°$$

b) $\cos R = 0.22$

$$\angle R = \cos^{-1}(0.22)$$
$$\angle R \doteq 77.3°$$
$$180° - \angle R = 180° - 77.3°$$
$$= 102.7°$$

c) $\tan S = 0.44$

$$\angle S = \tan^{-1}(0.44)$$
$$\angle S \doteq 23.7°$$
$$180° - \angle R = 180° - 23.7°$$
$$= 156.3°$$

Method 2

First determine the trigonometric ratio of the obtuse angle.

a) $\sin (180° - P) = \sin P$

$$= 0.65$$

Using a calculator:

$$\sin^{-1}(0.65) \doteq 40.5°$$
$$180° - 40.5° = 139.5°$$

b) $\cos (180° - R) = -\cos R$

$$= -0.22$$

Using a calculator:

$$\cos^{-1}(-0.22) \doteq 102.7°$$

c) $\tan (180° - S) = -\tan S$

$$= -0.44$$

Using a calculator:

$$\tan^{-1}(-0.44) \doteq -23.7°$$
$$180° + (-23.7°) = 156.3°$$

Since different angles have the same trigonometric ratios, your calculator may not return the measure of the obtuse angle when you use an inverse trigonometric ratio.

To determine the measure of an obtuse angle, A, using a calculator:

- $\angle A = 180° - \sin^{-1}(\sin A)$
- $\angle A = \cos^{-1}(\cos A)$
- $\angle A = 180° + \tan^{-1}(\tan A)$

See how these properties are applied in *Method 2*.

Practice

A

For help with question 1, see Example 1.

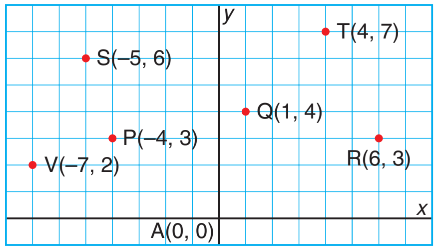
1. Determine the sine, cosine, and tangent ratios for each angle. Give each answer to 4 decimal places.

a) $110°$ **b)** $154°$ **c)** $102°$

2. Is each trigonometric ratio positive or negative? Use a calculator to check your answer.

a) $\sin 35°$ **b)** $\tan 154°$ **c)** $\cos 134°$

3. Each point on this coordinate grid makes a right triangle with the origin, A, and the x-axis. Determine the indicated trigonometric ratio in each triangle.



a) sin A and point P **b)** cos A and point Q
c) tan A and point R **d)** cos A and point S
e) sin A and point T **f)** tan A and point V

For help with questions 4, 5, and 6, see Example 2.

4. Suppose $\angle P$ is an obtuse angle. Determine the measure of $\angle P$ for each sine ratio. Give each angle measure to the nearest degree.

a) 0.23 **b)** 0.98 **c)** 0.57 **d)** 0.09

5. Determine the measure of $\angle R$ for each cosine ratio. Give each angle measure to the nearest degree.

a) -0.67 **b)** 0.56 **c)** -0.23 **d)** -0.25

6. Determine the measure of $\angle Q$ for each tangent ratio. Give each angle measure to the nearest degree.

a) 0.46 **b)** -1.60 **c)** -0.70 **d)** -1.53

B

7. $\angle M$ is between $0°$ and $180°$. Is $\angle M$ acute or obtuse? How do you know?

a) $\cos M = 0.6$ **b)** $\cos M = -0.6$ **c)** $\sin M = 0.6$

8. For each trigonometric ratio, identify whether $\angle Y$ could be between $0°$ and $180°$. Justify your answer.

a) $\cos Y = -0.83$ **b)** $\sin Y = -0.11$
c) $\tan Y = 0.57$ **d)** $\tan Y = 0.97$

9. **Literacy in Math** Create a table to organize information from this lesson about the sine, cosine, and tangent ratios of supplementary angles. Use *Connect the Ideas* to help you.

	Acute angle	Obtuse angle
Sine	positive	positive
Cosine		

10. $\angle G$ is an angle in a triangle. Determine all measures of $\angle G$.

a) $\sin G = 0.62$
b) $\cos G = -0.85$
c) $\tan G = 0.21$
d) $\tan G = -0.32$
e) $\cos G = -0.71$
f) $\sin G = 0.77$

11. The measure of $\angle Y$ is between 0° and 180°. Which equations result in two different values for $\angle Y$? How do you know?

a) $\sin Y = 0.32$
b) $\sin Y = 0.23$
c) $\cos Y = -0.45$
d) $\cos Y = 0.38$
e) $\tan Y = -0.70$
f) $\tan Y = 0.77$

12. **Assessment Focus** Determine all measures of $\angle A$ in a triangle, given each ratio. Explain your thinking.

a) $\sin A = 0.45$
b) $\cos A = -0.45$
c) $\tan A = 0.45$

13. The cosine of an obtuse angle is -0.45. Calculate the sine of this angle to 4 decimal places.

14. The sine of an obtuse angle is $\frac{12}{13}$. Calculate the cosine of this angle to 4 decimal places.

C

15. a) Determine the values of $\sin 90°$ and $\cos 90°$.

b) Explain why $\tan 90°$ is undefined.

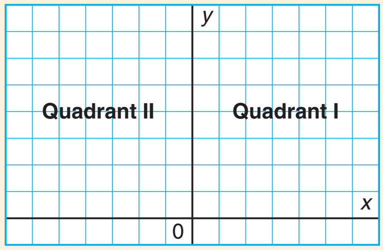
In Your Own Words

Is knowing a trigonometric ratio of an angle enough to determine the measure of the angle? If so, explain why. If not, what else do you need to know?

GAME

Trigonometric Search

Play with a partner.



- Each player:
  - Cuts out two 20 by 12 grids. Draws and labels each grid as shown.
  - Draws △ABC on the first grid. Writes the *x*- and *y*-coordinates of each vertex.
  - Joins vertices A, B, and C with the origin, O. Measures the angle made by each vertex with the positive *x*-axis.
  - Calculates a trigonometric ratio of their choice for the angle measured.
  - In turn, states the *x*- or *y*-coordinate and the calculated trigonometric ratio for the angle measured.
- The other player:
  - Uses the ratio to calculate the angle made with the positive *x*-axis.
  - Uses the *x*- or *y*-coordinate to plot the point.
  - Uses primary trigonometric ratios to find the second coordinate of the vertex.
  - Marks the findings on the second grid.

Player A:

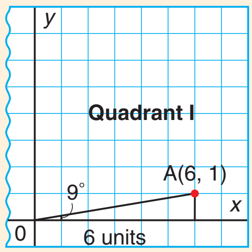
$\sin A = 0.16$
and $x = 6$ units

What are the coordinates of vertex A?

Player B:

- $\angle A = \sin^{-1}(0.16) \doteq 9°$
- $\tan 9° = \frac{y}{6}$
- $y = \tan 9° \times 6 \doteq 0.95$

So: A(6, 1)



The player who first solves the other player's triangle wins the round. Repeat the game with different triangles.

Materials

- grid paper
- scissors
- protractor
- scientific calculator

Rules for drawing △ABC

- Draw all vertices on intersecting grid lines.
- Do not draw any vertex on an axis.

Reflect

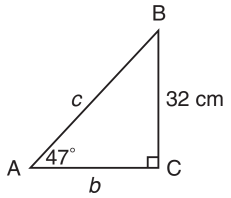
- What strategy did you use to determine the other coordinate of each vertex?
- What are the most common mistakes one can make during the game? How can you avoid them?

Mid-Chapter Review

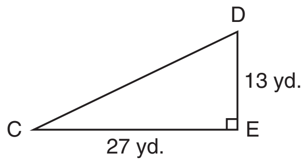
1.1

1. Solve each triangle.

a)



b)

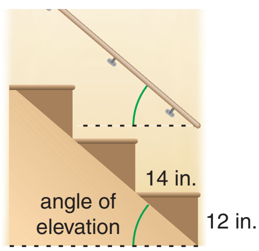


2. Solve each right $\triangle XYZ$. Sketch a diagram.

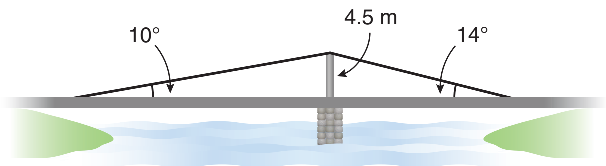
a) $\angle X = 53°$, $\angle Z = 90°$, $x = 3.6$ cm

b) $z = 3$ feet 5 inches, $x = 25$ inches, $\angle Z = 90°$

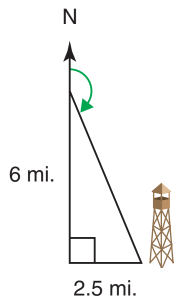
3. A flight of stairs has steps that are 14 inches deep and 12 inches high. A handrail runs along the wall in line with the steps. What is the angle of elevation of the handrail?



4. Cables stretch from each end of a bridge to a 4.5 m column on the bridge. The angles of elevation from each end of the bridge to the top of the column are 10° and 14°. What is the length of the bridge?



5. John hikes 2.5 miles due west from Temagami fire tower, then 6 miles due north.



a) How far is he from the fire tower at the end of the hike?

b) What bearing should he use to return to the fire tower?

Answer questions 6 and 7 without using a calculator.

1.2

6. Determine whether the sine, cosine, and tangent of the angle is positive or negative. Explain how you know.

a) 27° **b)** 95° **c)** 138°

7. Is $\angle P$ acute or obtuse? Explain.

a) $\cos P = 0.46$ **b)** $\tan P = -1.43$

c) $\sin P = 0.5$ **d)** $\cos P = -0.5877$

1.3

8. Determine each measure of obtuse $\angle P$.

a) $\sin P = 0.22$ **b)** $\cos P = -0.98$

c) $\tan P = -1.57$ **d)** $\sin P = 0.37$

9. $\angle G$ is an angle in a triangle. Determine all possible values for $\angle G$.

a) $\sin G = 0.53$ **b)** $\cos G = -0.42$

c) $\tan G = 0.14$ **d)** $\sin G = 0.05$

Home Quit

1.4 The Sine Law

The team of computer drafters working on the Michael Lee-Chin Crystal, at the Royal Ontario Museum, needed to know the precise measures of angles and sides in triangles that were not right triangles.

Investigate

Relating Sine Ratios in Triangles

Materials

- protractor
- scientific calculator

An acute triangle has three acute angles.
An obtuse triangle has one obtuse angle.

Work with a partner.

- Draw two large triangles: one acute and one obtuse.
  Label the vertices of each triangle A, B, and C.
- Copy and complete the table for each triangle.

Angle	Angle measure	Sine of angle	Length of opposite side	Ratios	
$\angle A$			$a =$	$\frac{a}{\sin A} =$	$\frac{\sin A}{a} =$
$\angle B$			$b =$	$\frac{b}{\sin B} =$	$\frac{\sin B}{b} =$
$\angle C$			$c =$	$\frac{c}{\sin C} =$	$\frac{\sin C}{c} =$

- Describe any relationships you notice in the tables.

Reflect

Compare your results with other pairs. Are the relationships true for all triangles? How does the *Investigate* support this?

Connect the Ideas

An **oblique triangle** is any triangle that is not a right triangle.

The Sine Law

The **Sine Law** relates the sides and the angles in any oblique triangle.

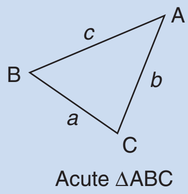
An acute triangle is an oblique triangle.
An obtuse triangle is an oblique triangle.

The Sine Law

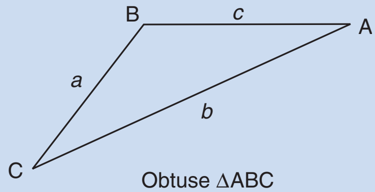
In any oblique △ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Acute △ABC



Obtuse △ABC

Angle-Angle-Side (AAS)

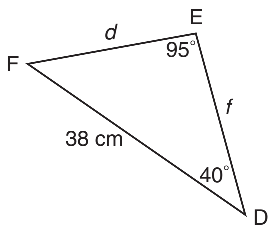
When we know the measures of two angles in a triangle and the length of a side opposite one of the angles, we can use the Sine Law to determine the length of the side opposite the other angle.

Example 1 Determining the Length of a Side Using the Sine Law

Materials

- scientific calculator

What is the length of side d in △DEF?



Solution

Write the Sine Law for △DEF:

$$\frac{d}{\sin D} = \frac{e}{\sin E} = \frac{f}{\sin F}$$

Use the two ratios that include the known measures.

$$\frac{d}{\sin D} = \frac{e}{\sin E}$$ Substitute: ∠D = 40°, ∠E = 95°, and $e = 38$

$$\frac{d}{\sin 40°} = \frac{38}{\sin 95°}$$ Multiply each side by sin 40° to isolate d.

$$d = \frac{38}{\sin 95°} \times \sin 40°$$

$$\doteq 24.52$$

Press: 38 ÷ SIN 95) × SIN 40) ENTER

So, side d is about 25 cm long.

Home Quit

Angle-Side-Angle (ASA)

When we know the measure of two angles in a triangle and the length of the side between them, we can determine the measure of the unknown angle using the sum of the angles in a triangle, then we can use the Sine Law to solve the triangle.

Example 2

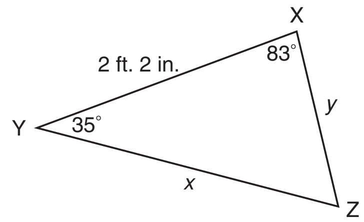
Materials

- scientific calculator

Determining the Measure of an Angle Using the Sine Law

a) Calculate the measure of $\angle Z$ in $\triangle XYZ$.

b) Determine the unknown side lengths x and y.



Solution

The sum of the angles in a triangle is 180°.

1 foot = 12 inches

a) $\angle Z = 180° - 83° - 35°$

$= 62°$

So, $\angle Z$ is 62°.

b) Write the length of side z in inches: 2 feet 2 inches = 26 inches

Write the Sine Law for $\triangle XYZ$: $\frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z}$

To find x, use the first and the third ratios.

$\frac{x}{\sin X} = \frac{z}{\sin Z}$ Substitute: $\angle X = 83°$, $\angle Z = 62°$, and $z = 26$

$\frac{x}{\sin 83°} = \frac{26}{\sin 62°}$ Multiply each side by sin 83° to isolate x.

$x = \frac{26}{\sin 62°} \times \sin 83°$

$\doteq 29.23$

So, side x is about 30 inches, or about 2 feet 6 inches, long.

To find y, use the second and third ratios in the Sine Law for $\triangle XYZ$.

$\frac{y}{\sin Y} = \frac{z}{\sin Z}$ Substitute: $\angle Y = 35°$, $\angle Z = 62°$, and $z = 26$

$\frac{y}{\sin 35°} = \frac{26}{\sin 62°}$ Multiply each side by sin 35° to isolate y.

$y = \frac{26}{\sin 62°} \times \sin 35°$

$\doteq 16.89$

So, side y is about 17 inches, or about 1 foot 5 inches, long.

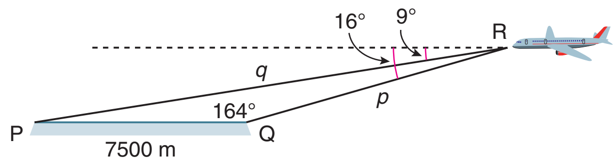
Example 3

Materials

- scientific calculator

Applying the Sine Law

A plane is approaching a 7500 m runway.
The angles of depression to the ends of the runway are 9° and 16°.
How far is the plane from each end of the runway?



Solution

To determine ∠R in △PQR, use the angles of depression.
$\angle PRQ = 16° - 9° = 7°$
Write the Sine Law for △PQR:
$$\frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}$$
To determine q, use the second and third ratios.

$$\frac{q}{\sin Q} = \frac{r}{\sin R}$$ Substitute: $\angle R = 7°$, $\angle Q = 164°$, and $r = 7500$

$$\frac{q}{\sin 164°} = \frac{7500}{\sin 7°}$$ Multiply each side by sin 164° to isolate q.

$$q = \frac{7500}{\sin 7°} \times \sin 164°$$
$$\doteq 16\,963.09$$

$$\angle P = 180° - 164° - 7°$$
$$= 9°$$

To determine p, use the first and third ratios in the Sine Law for △PQR.

$$\frac{p}{\sin P} = \frac{r}{\sin R}$$ Substitute: $\angle R = 7°$, $\angle P = 9°$, and $r = 7500$

$$\frac{p}{\sin 9°} = \frac{7500}{\sin 7°}$$ Multiply each side by sin 9° to isolate p.

$$p = \frac{7500}{\sin 7°} \times \sin 9°$$
$$\doteq 9627.18$$

So, the plane is about 16 963 m from one end and about 9627 m from the other end of the runway.

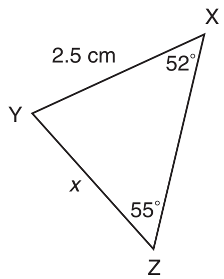
Practice

A

■ For help with questions 1 and 2, see Example 1.

1. Write the Sine Law for each triangle. Circle the ratios you would use to calculate each indicated length.

a)



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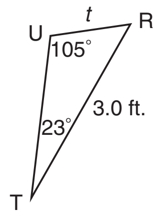
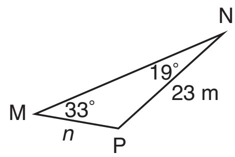


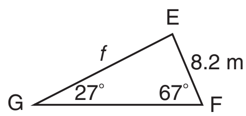
c)



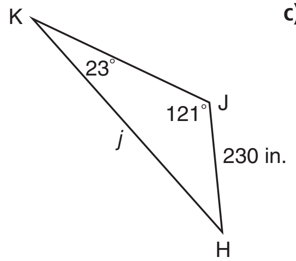
2. For each triangle in question 1, calculate each indicated length.

3. Use the Sine Law to determine the length of each indicated length.

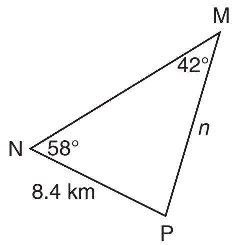
a)



b)



c)



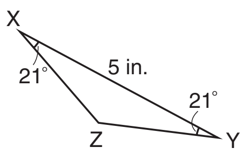
■ For help with questions 4 and 5, see Example 2.

4. **a)** Write the Sine Law for each triangle.

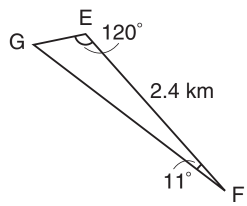
b) Circle the ratios you would use to find each unknown side.

c) Solve each triangle.

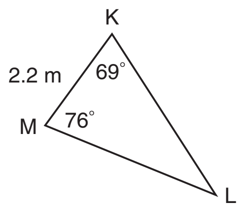
i)



ii)

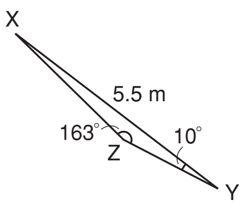


iii)



5. Solve each triangle.

a)



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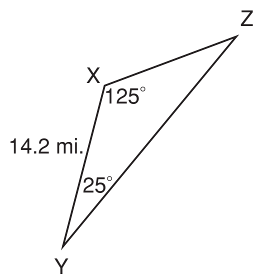

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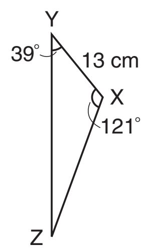
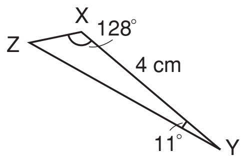


6. Solve each triangle.

a)



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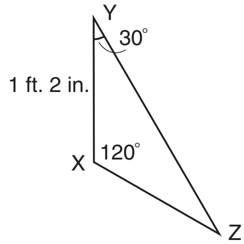

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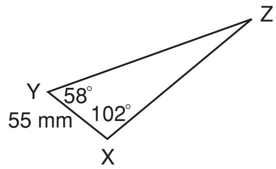
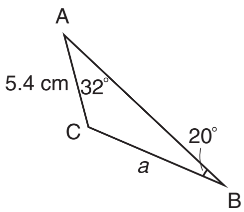


B

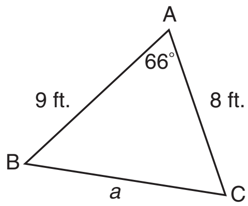
7. Choose one part from question 6. Write to explain how you solved the triangle.

8. Could you use the Sine Law to determine the length of side *a* in each triangle? If not, explain why not.

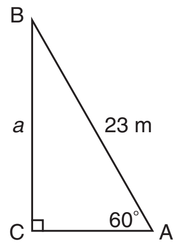
a)



b)



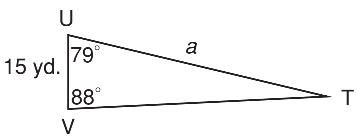
c)



9. In question 8, determine *a* where possible using the Sine Law.

10. a) What is the measure of ∠T in △TUV?

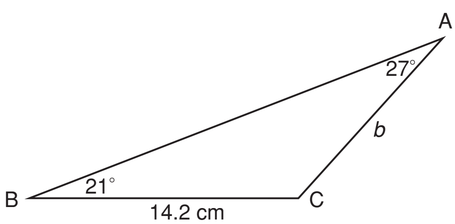
b) Determine the length of side *a*.



11. a) Use the Sine Law to determine the length of side *b*.

b) Explain why you would use the Sine Law to solve this problem.

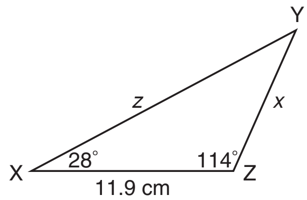
c) Which pair of ratios did you use to solve for *b*? Explain your choice.



12. In $\triangle XYZ$:

a) Determine the lengths of sides x and z.

b) Explain your strategy.



13. In $\triangle DEF$, $\angle D = 29°$, $\angle E = 113°$, and $f = 4.2$ inches.

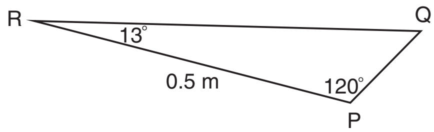
a) Determine the length of side e.

b) Explain how you solved the problem.

14. **Literacy in Math** Describe the steps for solving question 13. Draw a diagram to show your work.

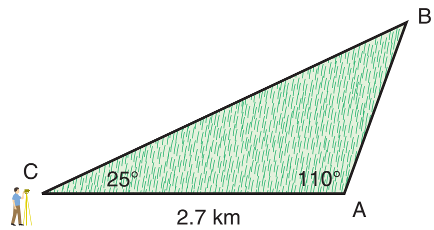
For help with question 15, see Example 3.

15. A welder needs to cut this triangular shape from a piece of metal.



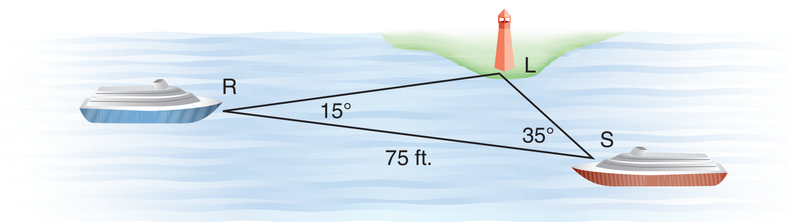
Determine the measure of $\angle Q$ and the side lengths PQ and QR.

16. **Assessment Focus** A surveyor is mapping a triangular plot of land. Determine the unknown side lengths and angle measure in the triangle. Describe your strategy.



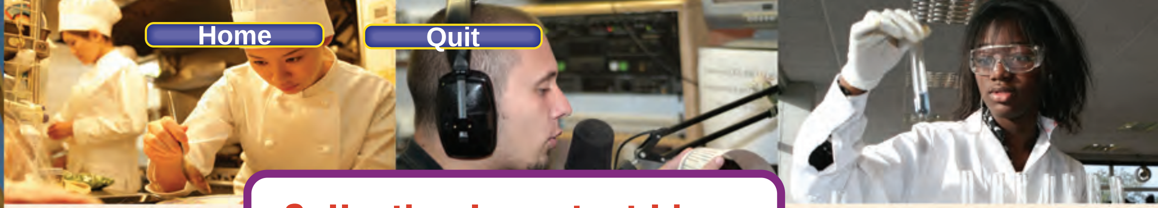
C

17. How far are ships R and S from lighthouse L?



In Your Own Words

What information do you need in a triangle to be able to use the Sine Law? How do you decide which pair of ratios to use?



Collecting Important Ideas

Focusing on key ideas can help you improve performance at both work and school. As you prepare for college or a career, keep track of the important ideas you learn.

When recording important ideas:

- Use captions, arrows, and colours to help show the idea.
- Add a picture, a diagram, or an example.
- Keep it brief.
  Don’t say more than you need to!

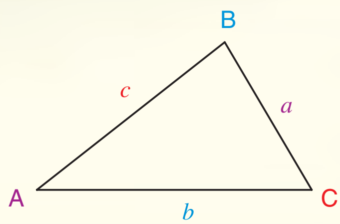
Here’s an example.

The Sine Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

You need one pair and one more side or angle.



1. What changes would you make to the example above?

2. Begin with this chapter.
Make a collection of “trigonometry” ideas.
Create a section in your math notebook or start a file on your computer.

3. During the year, add to your collection by creating a record of important ideas from later chapters.

Home Quit

1.5 The Cosine Law

A furniture designer's work begins with a concept, developed on paper or on a computer, then built to test its practicality and functionality.
The designs may then be mass-produced. Precise angle measurements are required at all times.

Investigate — Using Cosine Ratios in Triangles

Materials
- protractor
- scientific calculator

Work with a partner.

- Construct a $\triangle ABC$ for each description.
  - $\angle C = 90°$
  - All angles are acute
  - $\angle C$ is obtuse
- Copy and complete the table.

$\triangle ABC$	a	a^2	b	b^2	c	c^2	$\cos C$	$2ab \cos C$	$a^2+b^2-2ab \cos C$
Right triangle									
Acute triangle									
Obtuse triangle									

- What patterns do you notice in the table?
- Compare your results with other pairs. Are your results true for all triangles?

Reflect

The Cosine Law is: $c^2 = a^2 + b^2 - 2ab \cos C$.
How are the Cosine Law for triangles and the Pythagorean Theorem the same? How are they different? How does the information in your table show this?

Connect the Ideas

The Cosine Law

The Sine Law, although helpful, has limited applications. We cannot use the Sine Law to solve an oblique triangle unless we know the measures of at least two angles. We need another method when:

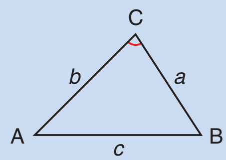
- We know the lengths of two sides and the measure of the angle between them (SAS)
- We know all three side lengths in a triangle (SSS)

The **Cosine Law** can be used in both of these situations.

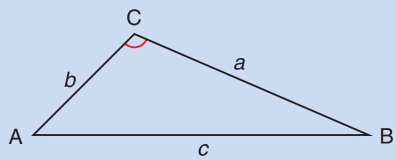
The Cosine Law

In any oblique $\triangle ABC$:

$c^2 = a^2 + b^2 - 2ab \cos C$



Acute triangle



Obtuse triangle

Side-Angle-Side (SAS)

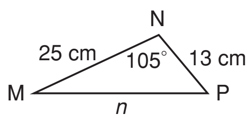
In any triangle, given the lengths of two sides and the measure of the angle between them, we can use the Cosine Law to determine the length of the third side.

Example 1 Determining a Side Length Using the Cosine Law

Materials

- scientific calculator

Determine the length of n in $\triangle MNP$.



Solution

The Sine Law cannot be used here, because no pair of ratios includes the three given measures, m, p, and $\angle N$.

Write the Cosine Law for $\triangle MNP$.

$n^2 = m^2 + p^2 - 2mp \cos N$ Substitute: $m = 13$, $p = 25$, and $\angle N = 105°$

$= 13^2 + 25^2 - 2 \times 13 \times 25 \times \cos 105°$

$\doteq 962.2323$

$n = \sqrt{962.2323}$

$\doteq 31.02$

n is about 31 cm long.

Apply the order of operations.

Press: 13 $\boxed{x^2}$ $\boxed{+}$ 25 $\boxed{x^2}$ $\boxed{-}$ 2 $\boxed{\times}$ 13 $\boxed{\times}$ 25 $\boxed{\times}$ $\boxed{\text{COS}}$ 105 $\boxed{)}$ $\boxed{\text{ENTER}}$

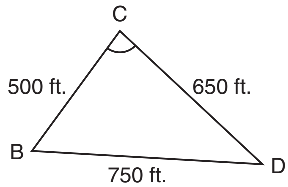
Side-Side-Side (SSS)

We can use the Cosine Law to calculate the measure of an angle in a triangle when the lengths of all three sides are known.

Example 2 Determine an Angle Measure Using the Cosine Law

Materials
- scientific calculator

Determine the measure of ∠C in △BCD.



Solution

Write the Cosine Law for △BCD using ∠C.

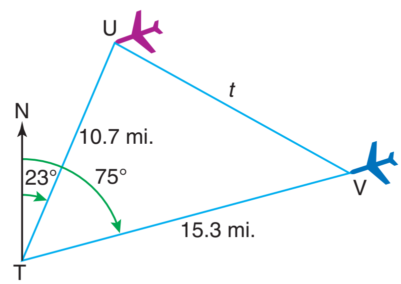
$$c^2 = b^2 + d^2 - 2bd \cos C \quad \text{Substitute: } c = 750, b = 650, d = 500$$
$$750^2 = 650^2 + 500^2 - 2 \times 650 \times 500 \times \cos C$$
$$562\,500 = 422\,500 + 250\,000 - 650\,000 \times \cos C$$
$$-110\,000 = -650\,000 \times \cos C \quad \text{Isolate } \cos C.$$
$$\cos C = \frac{-110\,000}{-650\,000} \quad \text{Isolate C.}$$
$$\angle C = \cos^{-1}\left(\frac{-110\,000}{-650\,000}\right)$$
$$\angle C \doteq 80.2569$$

∠C is about 80°.

Example 3 Navigating Using the Cosine Law

Materials
- scientific calculator

An air-traffic controller at T is tracking two planes, U and V, flying at the same altitude. How far apart are the planes?



Solution

$\angle UTV = 75° - 23° = 52°$

Write the Cosine Law for △TUV using ∠T.

$$t^2 = u^2 + v^2 - 2uv \cos T \quad \text{Substitute: } u = 15.3, v = 10.7, \angle T = 52°$$
$$= 15.3^2 + 10.7^2 - 2 \times 15.3 \times 10.7 \times \cos 52°$$
$$\doteq 147.0001$$
$$t = \sqrt{147.0001}$$
$$\doteq 12.12$$

The planes are about 12.1 miles apart.

The number of decimal places in your answer should match the given measures.

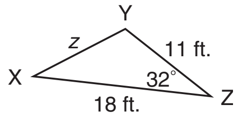
Practice

A

■ For help with questions 1 and 2, see Example 1.

1. Write the Cosine Law you would use to determine each indicated side length.

a)



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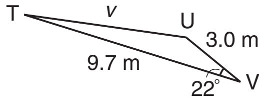

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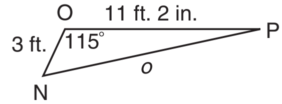
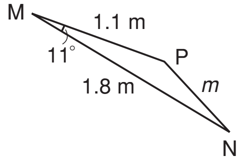


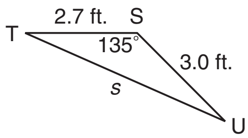
2. Determine each unknown side length in question 1.

3. Determine each unknown side length.

a)



b)



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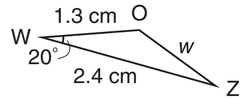
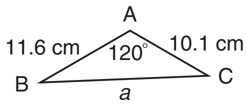


4. Determine the length of *a*.

a)



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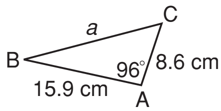

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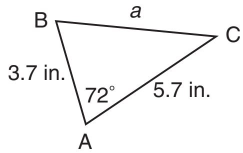
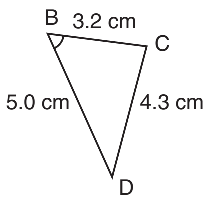


■ For help with questions 5 and 6, see Example 2.

5. Write the Cosine Law you would use to determine the measure of the marked angle in each triangle.

a)



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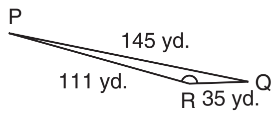

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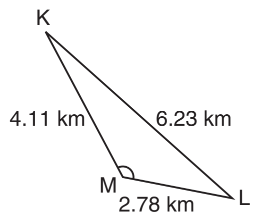
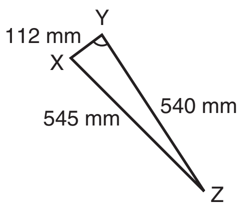


6. Determine the measure of each marked angle in question 5.

7. Determine the measure of each unknown angle.

a)



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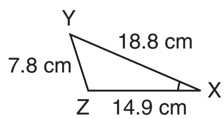
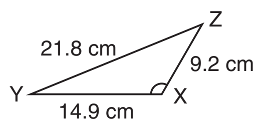


c)



8. Determine the measure of each unknown angle.
Start by sketching the triangle.

a) In $\triangle XYZ$, $z = 18.8$ cm, $x = 24.8$ cm, and $y = 9.8$ cm
Determine $\angle X$.

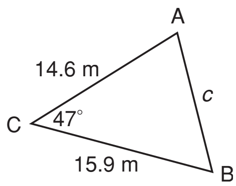
b) In $\triangle GHJ$, $g = 23$ feet, $h = 25$ feet, and $j = 31$ feet
Determine $\angle J$.

c) In $\triangle KLM$, $k = 12.9$ yards, $l = 17.8$ yards, $m = 14.8$ yards
Determine $\angle M$.

B

9. Would you use the Cosine Law or the Sine Law to determine the labelled length in each triangle? Explain your reasoning.

a)



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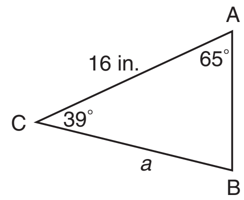

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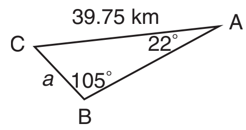
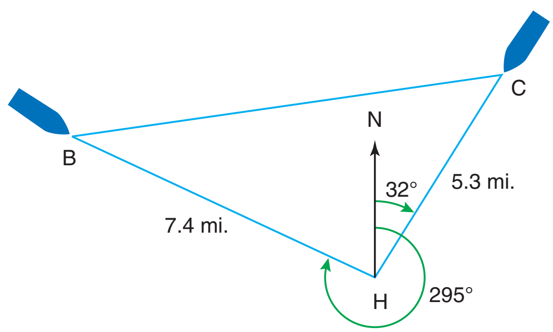


■ For help with questions 10 and 11, see Example 3.

10. A harbour master uses a radar to monitor two ships, B and C, as they approach the harbour, H. One ship is 5.3 miles from the harbour on a bearing of 032°. The other ship is 7.4 miles away from the harbour on a bearing of 295°.

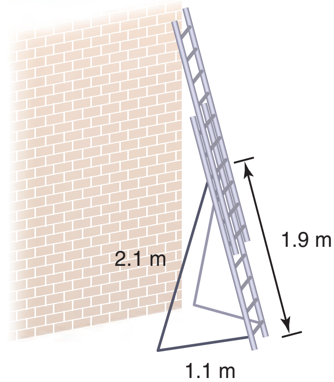
a) How are the bearings shown in the diagram?

b) How far apart are the two ships?

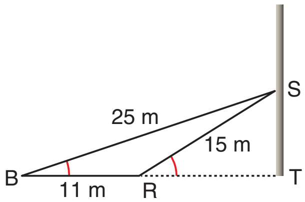


11. A theatre set builder's plans show a triangular set with two sides that measure 3 feet 6 inches and 4 feet 9 inches. The angle between these sides is 45°. Determine the length of the third side.

12. A telescoping ladder has a pair of aluminum struts, called ladder stabilizers, and a base. What is the angle between the base and the ladder?



13. A hydro pole needs two guy wires for support. What angle does each wire make with the ground?



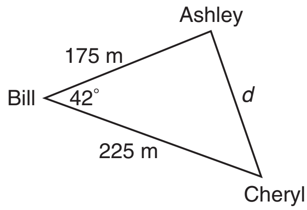
14. A land survey shows that a triangular plot of land has side lengths 2.5 miles, 3.5 miles, and 1.5 miles. Determine the angles in the triangle. Explain how this problem could be done in more than one way.

15. **Assessment Focus** An aircraft navigator knows that town A is 71 km due north of the airport, town B is 201 km from the airport, and towns A and B are 241 km apart.

a) On what bearing should she plan the course from the airport to town B? Include a diagram.

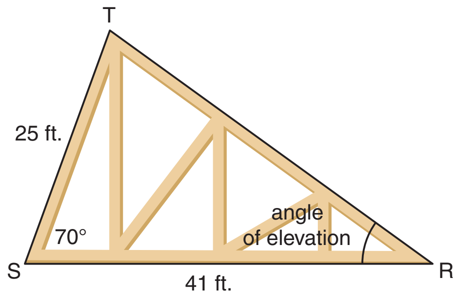
b) Explain how you solved the problem.

16. **Literacy in Math** Write a problem that can be represented by this diagram. Solve your problem.



C

17. Use this diagram of a roof truss.

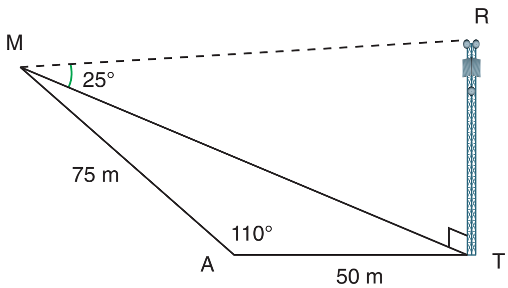


a) Determine the length of TR.

b) What is the angle of elevation? Record your answer to the nearest degree.

Indirect measurements are used to determine inaccessible distances and angles, which cannot be measured directly.

18. Marie wants to determine the height of an Internet transmission tower. Due to several obstructions, she has to use indirect measurements to determine the tower height. She walks 50 m from the base of the tower, turns 110°, and then walks another 75 m. Then she measures the angle of elevation to the top of the tower to be 25°.



a) What is the height of the tower?

b) What assumptions did you make? Do you think it is reasonable to make these assumptions? Justify your answer.

In Your Own Words

How can the Pythagorean Theorem help you remember the Cosine Law? Include a diagram with your explanation.

Home Quit

1.6 Problem Solving with Oblique Triangles

Campfires, a part of many camping experiences, can be dangerous if proper precautions are not taken. Forest rangers advise campers to light their fires in designated locations, away from trees, tents, or other fire hazards.

Investigate

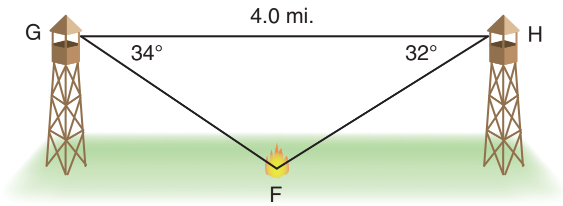
Choosing Sine Law or Cosine Law

Materials

- protractor
- scientific calculator

Work with a partner.

Two forest rangers sight a campfire, F, from their observation towers, G and H. How far is the fire from each observation deck?



Decide whether to use the Sine Law or the Cosine Law to determine each distance.

Reflect

- Compare strategies with another pair. How are they different? How are they the same?
- How did you decide which law or laws to use? Justify your choices.
- Could you have used another law? Explain why or why not.

Connect the Ideas

Sine Law

In any triangle, we can use the Sine Law to solve the triangle when:
- We know the measures of two angles and the length of any side (AAS or ASA)

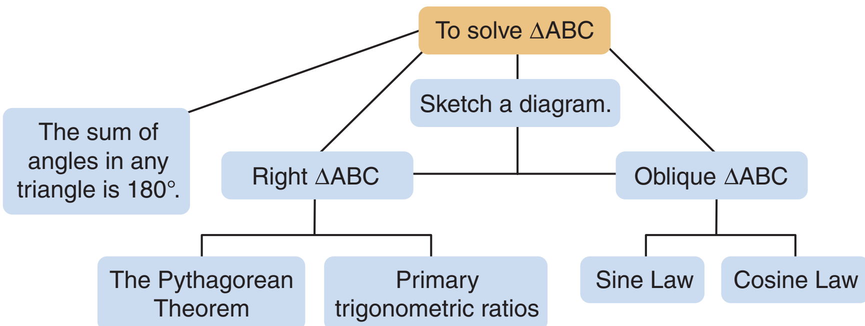
Cosine Law

In any triangle, we can use the Cosine Law to solve the triangle when:
- We know the lengths of two sides and the measure of the angle between them (SAS)
- We know the lengths of three sides (SSS)

Solving triangle problems

When solving triangle problems:
- Read the problem carefully.
- Sketch a diagram if one is not given. Record known measurements on your diagram.
- Identify the unknown and known measures.
- Use a triangle relationship to determine the unknown measures.

Often, two or more steps may be needed to solve a triangle problem.

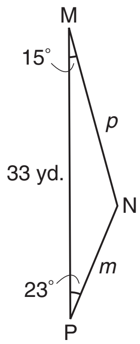


Example 1 Determining Side Lengths

Materials

- scientific calculator

Decide whether to use the Sine Law or Cosine Law to determine m and p in $\triangle$MNP. Then, determine each side length.



Solution

Since $\triangle$MNP is an oblique triangle, we need to use the Sine Law or the Cosine Law to solve it.

In oblique $\triangle$MNP, we know the measures of two angles and the length of the side between them (ASA). So, use the Sine Law.

Write the Sine Law for $\triangle$MNP:

$$\frac{m}{\sin M} = \frac{n}{\sin N} = \frac{p}{\sin P}$$

$$\angle N = 180° - 15° - 23° = 142°$$

To find m, use the first two ratios.

$$\frac{m}{\sin M} = \frac{n}{\sin N}$$ Substitute: $n = 33$, $\angle M = 15°$, and $\angle N = 142°$

$$\frac{m}{\sin 15°} = \frac{33}{\sin 142°}$$ Multiply both sides by sin 15° to isolate m.

$$m = \frac{33}{\sin 142°} \times \sin 15°$$

$$m \doteq 13.87$$

m is about 14 yards.

To find p, use the second and third ratios.

$$\frac{n}{\sin N} = \frac{p}{\sin P}$$ Substitute: $n = 33$, $\angle N = 142°$, and $\angle P = 23°$

$$\frac{33}{\sin 142°} = \frac{p}{\sin 23°}$$ Multiply both sides by sin 23° to isolate p.

$$p = \frac{33}{\sin 142°} \times \sin 23°$$

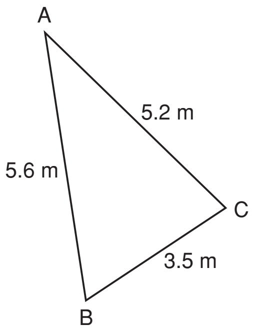
$$p \doteq 20.94$$

p is about 21 yards.

Example 2 — Determining Angle Measures

Materials
- scientific calculator

Write the equation you would use to determine the cosine of each angle in $\triangle$ABC. Determine each angle measure.



Solution

In oblique $\triangle$ABC, we are given the lengths of all three sides (SSS). So, use the Cosine Law.

Write the Cosine Law using $\angle$C:

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{Isolate cos C.}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{Isolate C.}$$

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \qquad \text{Substitute: } a = 3.5,\ b = 5.2,\ c = 5.6$$

$$= \cos^{-1}\left(\frac{3.5^2 + 5.2^2 - 5.6^2}{2 \times 3.5 \times 5.2}\right)$$

$$\doteq 77.42°$$

So, the measure of $\angle$C is about 77°.

Write the Cosine Law using $\angle$B:

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad \text{Isolate cos B.}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \text{Isolate B.}$$

$$B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) \qquad \text{Substitute: } a = 3.5,\ b = 5.2,\ c = 5.6$$

$$= \cos^{-1}\left(\frac{3.5^2 + 5.6^2 - 5.2^2}{2 \times 3.5 \times 5.6}\right)$$

$$\doteq 64.99°$$

So, the measure of $\angle$B is about 65°.

$\angle A = 180° - 77° - 65° = 38°$

So, the measure of $\angle$A is about 38°.

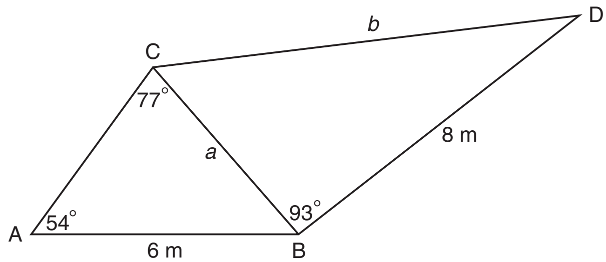
Example 3

Materials

- scientific calculator

Solving Two Oblique Triangles

Determine lengths a and b.



Solution

In oblique $\triangle$ABC, we know the measures of two angles and the length of the side opposite one of them (AAS). So, use the Sine Law.

Write the Sine Law for $\triangle$ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

To find a, use the first and the third ratios.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$ Substitute: $\angle C = 77°$, $\angle A = 54°$, and $c = 6$

$$\frac{a}{\sin 54°} = \frac{6}{\sin 77°}$$ Multiply both sides by sin 54° to isolate a.

$$a = \frac{6}{\sin 77°} \times \sin 54°$$

$$\doteq 4.98$$

So, a is about 5 m long.

In oblique $\triangle$BCD, we now know the lengths of two sides and the measure of the angle between them (SAS). So, use the Cosine Law.

Write the Cosine Law for $\triangle$BCD using $\angle$B.

$$b^2 = a^2 + c^2 - 2ac \cos B$$ Substitute: $a = 4.98$, $c = 8$, $\angle B = 93°$

$$= 4.98^2 + 8^2 - 2 \times 4.98 \times 8 \times \cos 93°$$

$$\doteq 92.97$$

$$b = \sqrt{92.97}$$

$$\doteq 9.64$$

So, b is about 10 m long.

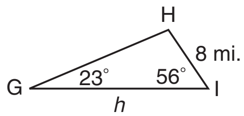
To ensure accurate results when calculating b, use the value for a before rounding.

Practice

A

■ For help with questions 1 and 2, see Example 1.

1. Decide whether you would use the Sine Law or the Cosine Law to determine each indicated length.

a) 

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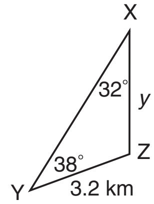
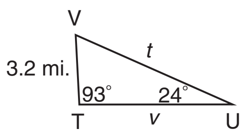
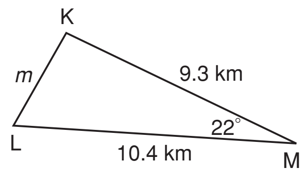
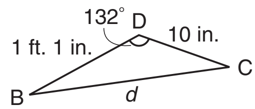
c) 

2. Determine each indicated length in question 1.

3. Use the Sine Law or the Cosine Law to determine each indicated length.

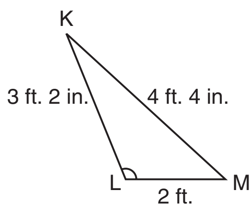
a) 

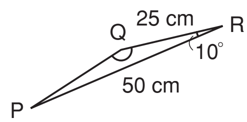
b) 

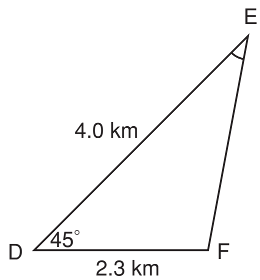
c) 

■ For help with questions 4 and 5, see Example 2.

4. Decide whether you would use the Sine Law or the Cosine Law to determine each angle measure.

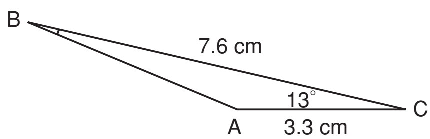
a) 

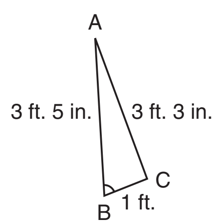
b) 

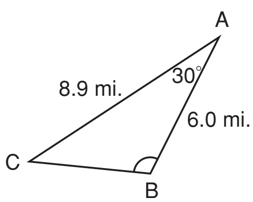
c) 

5. Determine the measure of each angle in question 4.

6. Use the Sine Law or the Cosine Law to determine each measure for ∠B.

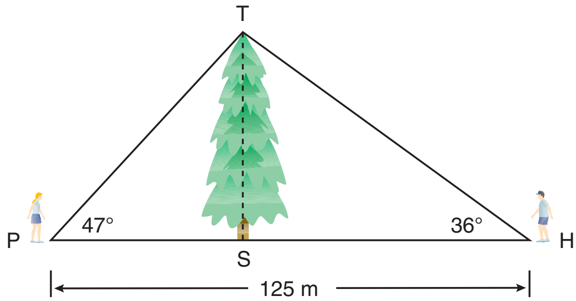
a) 

b) 

c) 

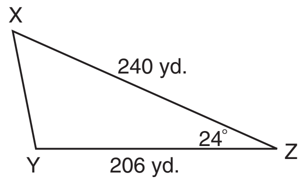
■ For help with questions 7 and 8, see Example 3.

7. Phoebe and Holden are on opposite sides of a tall tree, 125 m apart. The angles of elevation from each to the top of the tree are 47° and 36°. What is the height of the tree?



B

8. Carrie says she can use the Cosine Law to solve △XYZ. Do you agree? Justify your answer.



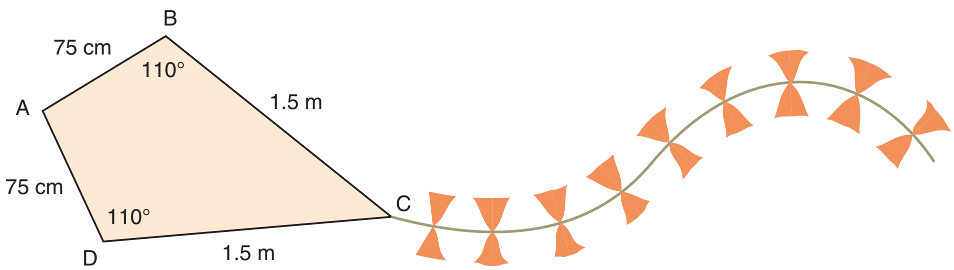
9. Use what you know from question 8. Write your own question that can be solved using the Sine Law or the Cosine Law. Show your solution.

The sum of the angles in a quadrilateral is 360°.

10. A hobby craft designer is designing this two-dimensional kite.

a) What is the angle measure between the longer sides?

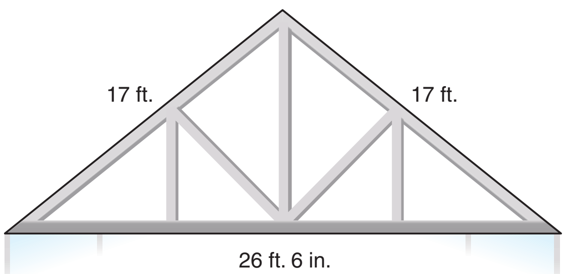
b) What is the angle measure between the shorter sides? Explain your strategies.



11. Use this diagram of the rafters in a greenhouse.

a) What angle do the rafters form at the peak of the greenhouse?

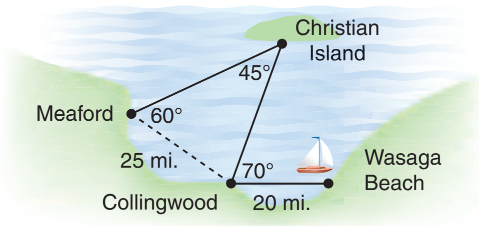
b) What angle do they form with the sides of the greenhouse? Solve this problem two ways: using the Cosine Law and using primary trigonometric ratios.



12. A boat sails from Meaford to Christian Island, to Collingwood, then to Wasaga Beach.

a) What is the total distance the boat sailed?

b) What is the shortest distance from Wasaga Beach to Christian Island?



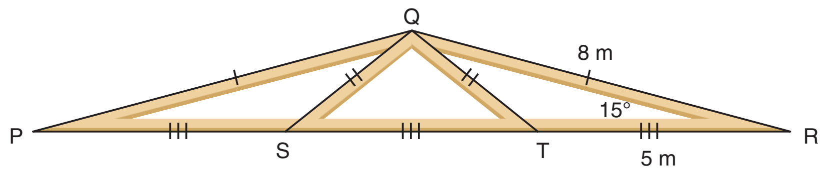
13. A triangle has side lengths measuring 5 inches, 10 inches, and 7 inches. Determine the angles in the triangle.

14. **Assessment Focus** Roof rafters and truss form oblique △PQR and △SQT.

a) Describe two different methods that could be used to determine ∠SQT.

b) Determine ∠SQT, using one of the methods described in part a.

c) Explain why you chose the method you used in part b.



15. **Literacy in Math** Explain the flow chart in *Connect the Ideas* in your own words. Add any other important information.

C **16.** Two boats leave port at the same time. One sails at 30 km/h on a bearing of 305°. The other sails at 27 km/h on a bearing of 333°. How far apart are the boats after 2 hours?

In Your Own Words

What mistakes can someone make while solving problems with the Sine Law and the Cosine Law? How can they be avoided?

1.7 Occupations Using Trigonometry

Professional tools, such as tilt indicators on boom trucks, lasers, and global positioning systems (GPS) perform trigonometric computations automatically.

Inquire

Researching Applications of Trigonometry

Materials
- computers with Internet access

It may be helpful to invite an expert from a field that uses trigonometry or an advisor from a college or apprenticeship program.

Work in small groups.

Part A: Planning the Research

- Brainstorm a list of occupations that involve the use of trigonometry. Include occupations you read about during this chapter.
- Briefly describe how each occupation you listed involves the use of trigonometry. Use these questions to guide you.
  - What measures and calculations might each occupation require?
  - What tools and technologies might each occupation use for indirect measurements? How do these tools and technologies work?
- Find sources about career guidance. Write some information that might be of interest to you or to someone you know.

Part B: Gathering Information

- Choose one occupation from your list. Investigate as many different applications of trigonometry as you can in the occupation you chose. Include answers to questions such as:
  - What measurement system is used: metric, imperial, or both?
  - How important is the accuracy of measurements and calculations?
  - What types of communication are used: written, graphical, or both?
  - What other mathematics does this occupation involve?
  - What is a typical wage for an entry-level position in this career?
  - What is a typical wage for someone with experience in this occupation?
  - Are employees paid on an hourly or a salary basis?

Think of any key words or combinations of key words that might help you with your research on the Internet.

- Read about the occupation you chose on the Internet or in printed materials.
- If possible, interview people who work in the occupation you chose. Prepare a list of questions you would ask them about trigonometry.

Search words
- occupational information
- working conditions
- other qualifications
- earnings, salaries
- related occupations

Search words

- training
- education information
- apprenticeship programs

Part C: Researching Educational Requirements

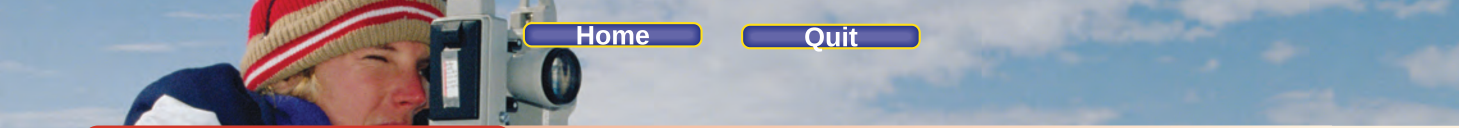
- Research to find out the educational requirements for the occupation you chose. On the Internet, use search words related to education. You might decide to use the same words you used before.
  - Find out about any pre-apprenticeship training programs or apprenticeship programs available locally.
  - Go to Web sites of community colleges, or other post-secondary institutions.
  - Use course calendars of post-secondary institutions or other information available through your school's guidance department.
- How can you get financial help for the required studies? How could you find out more?

Part D: Presenting Your Findings

- Prepare a presentation you might give your class, another group, or someone you know. Think about how you can organize and clearly present the information and data you researched. Use diagrams or graphic organizers if they help.

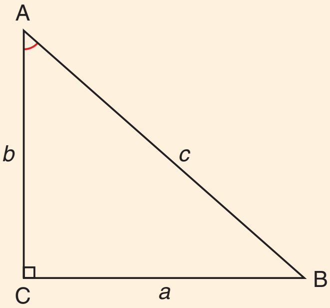
Reflect

- Write about a problem you might encounter in the occupation you researched. What would you do to solve it? How does your solution involve trigonometry?
- Why do you think it is important to have a good understanding of trigonometry in the occupation you researched?



Study Guide

Primary Trigonometric Ratios



When $\angle$A is an acute angle in a right $\triangle$ABC:

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{a}{c}$$

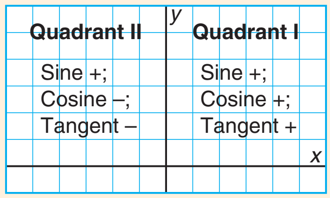
$$\cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{a}{b}$$

The inverse ratios are $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

- $\sin^{-1}(\sin A) = A$
- $\cos^{-1}(\cos A) = A$
- $\tan^{-1}(\tan A) = A$

Trigonometric Ratios of Supplementary Angles

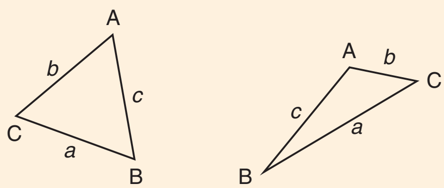


Two angles are supplementary if their sum is 180°.

For an acute angle, A, and its supplementary obtuse angle (180° − A):

- $\sin A = \sin(180° - A)$
- $\cos A = -\cos(180° - A)$
- $\tan A = -\tan(180° - A)$

The Sine Law



In any $\triangle$ABC: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

To use the Sine Law, you must know at least one side-angle pair (for example, *a* and $\angle$A).

To determine a side length, you must know an additional angle measure:

- angle–angle–side (AAS)
- angle–side–angle (ASA)

The Cosine Law

In any $\triangle$ABC: $c^2 = a^2 + b^2 - 2ab\cos C$

To determine a side length using the Cosine Law, you must know:

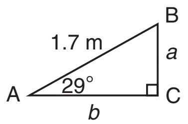
- side–angle–side (SAS)

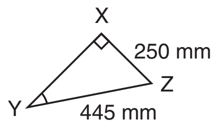
To determine an angle measure using the Cosine Law, you must know:

- side–side–side (SSS)

Chapter Review

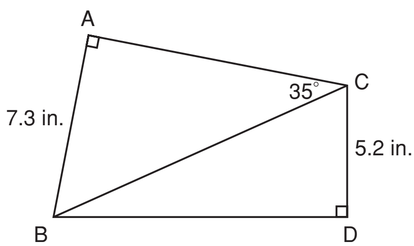
1.1 **1.** Determine each indicated measure.

a) 

b) 

2. Danny is building a ski jump with an angle of elevation of 15° and a ramp length of 4.5 m. How high will the ski jump be?

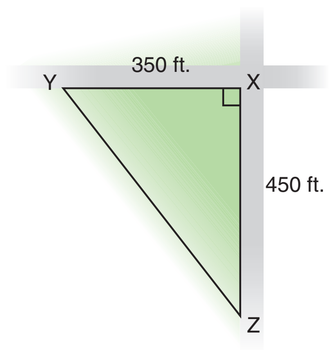
3. Determine the measure of ∠CBD.



4. A triangular lot is located at the intersection of two perpendicular streets. The lot extends 350 feet along one street and 450 feet along the other street.

a) What angle does the third side of the lot make with each road?

b) What is the perimeter of the lot? Explain your strategy.



5. A ship navigator knows that an island harbour is 20 km north and 35 km west of the ship's current position. On what bearing could the ship sail directly to the harbour?

Answer questions 6 to 8 without using a calculator.

1.2 **6.** Is each trigonometric ratio positive or negative? Explain how you know.

a) tan 53° b) cos 96° c) sin 132°

7. Is ∠B acute or obtuse? Explain.

a) tan B = 1.6 b) cos B = −0.9945

c) cos B = 0.35 d) sin B = 0.7

8. a) cos A = −0.94
Determine cos (180° − A).

b) sin A = 0.52
Determine sin (180° − A).

c) tan A = 0.37
Determine tan (180° − A).

1.3 **9.** Determine the measure of each obtuse angle.

a) sin R = 0.93 b) cos D = −0.56

10. 0° < ∠G < 180°. Is ∠G acute or obtuse? How do you know?

a) tan G = −0.2125 b) sin G = 0.087

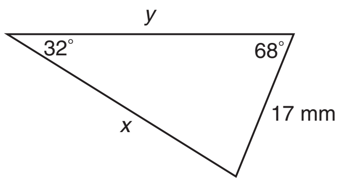
11. Determine the measure of ∠A. If you get more than one answer for the measure of ∠A, explain why.

a) tan A = −0.1746 b) sin A = 0.3584

12. ∠Z is an angle in a triangle. Determine all possible values for ∠Z.

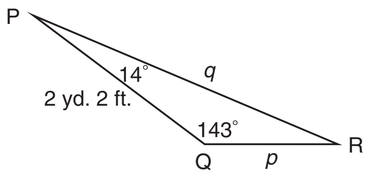
a) cos Z = 0.93 b) sin Z = 0.73

1.4 **13.** Use the Sine Law to determine x and y.



14. a) Determine the lengths of sides p and q.

b) Explain your strategy.



15. Solve each triangle.
Sketch a diagram first.

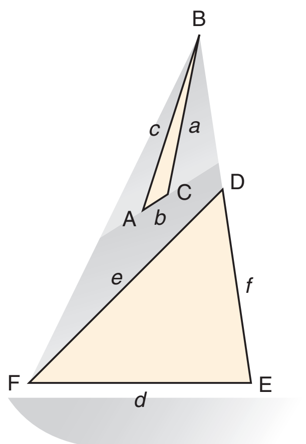
a) $\triangle LMN$, $\angle M = 48°$, $\angle N = 105°$, and $l = 17$ m

b) $\triangle HIJ$, $\angle H = 21°$, $\angle J = 57°$, and $h = 9$ feet 4 inches

16. Lani received these specifications for two different triangular sections of a sailboat sail.

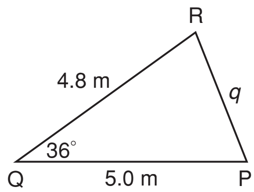
a) $a = 5.5$ m, $b = 1.0$ m, and $\angle C = 134°$. Determine $\angle A$.

b) $d = 7.75$ m, $e = 9.25$ m, and $\angle F = 45°$. Determine $\angle E$.



17. One side of a triangular lot is 2.6 m long. The angles in the triangle at each end of the 2.6-m side are 38° and 94°. Determine the lengths of the other two sides of the lot.

1.5 **18. a)** Explain why you would use the Cosine Law to determine q.

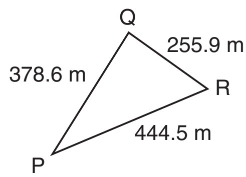


b) Which version of the Cosine Law would you use to solve this problem? Explain how you know the version you chose is correct.

c) Determine the length of side q.

19. Sketch and label $\triangle BCD$ with $b = 7.5$ km, $d = 4.3$ km, and $\angle C = 131°$. Solve $\triangle BCD$.

20. Determine the measure of $\angle Q$.



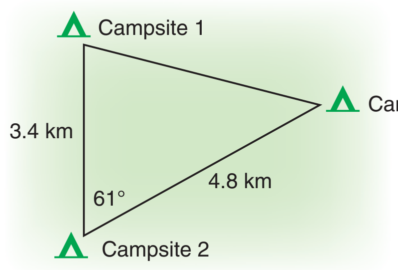
21. a) Sketch each triangle.

b) Determine the measure of the specified angle.

i) In $\triangle MNO$, $m = 3.6$ m, $n = 10.7$ m, and $o = 730$ cm. Determine $\angle N$.

ii) In $\triangle CDE$, $c = 66$ feet, $d = 52$ feet, and $e = 59$ feet. Determine $\angle D$.

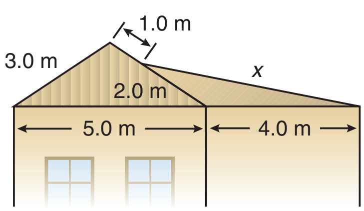
22. Jane is drawing an orienteering map that shows the location of three campsites. Determine the two missing angle measures.



23. A machinist is cutting out a large triangular piece of metal to make a part for a crane. The sides of the piece measure 4 feet 10 inches, 3 feet 10 inches, and 5 feet 2 inches. What are the angles between the sides?

1.6 **24.** Renée and Andi volunteered to help scientists measure the heights of trees in old-growth forests in Algonquin Park. The two volunteers are 20 m apart on opposite sides of an aspen. The angle of elevation from one volunteer to the top of the tree is 65°, and from the other, 75°. What is the height of the tree? What assumptions did you make?

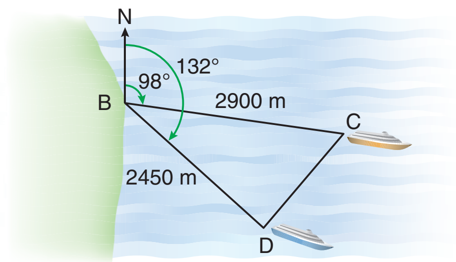
25. Determine the length, *x*, of the lean-to roof attached to the side of the cabin.



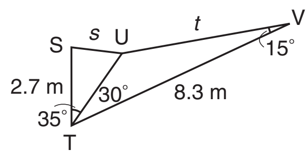
26. Two ferries leave dock B at the same time. One travels 2900 m on a bearing of 098°. The other travels 2450 m on a bearing of 132°.

a) How are the bearings shown on the diagram?

b) How far apart are the ferries?



27. Determine the lengths of sides *s* and *t*.



28. a) Write a word problem that can be solved using the Sine Law. Explain your strategy.

b) Write a word problem that can be solved with the Cosine Law. Explain your strategy.

1.7 **29.** Describe one occupation that uses trigonometry. Give a specific example of a calculation a person in this occupation might perform as part of their work.

30. Tell about an apprenticeship program or post-secondary institution that offers the training required by a person in the career you identified in question 29.

Practice Test

Multiple Choice: Choose the correct answer for questions 1 and 2. Justify each choice:

1. Which could you use to determine the measure of an angle in an oblique triangle if you only know the lengths of all three sides?

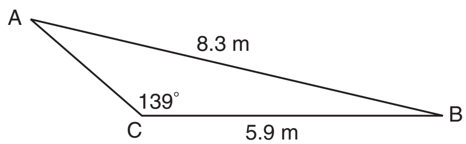
A. Sine Law **B.** Cosine Law **C.** Tangent ratio **D.** Sine ratio

2. If ∠A is obtuse, which is positive?

A. sin A **B.** cos A **C.** tan A **D.** none

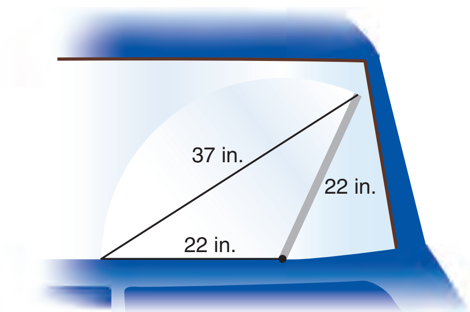
Show all your work for questions 3 to 6.

3. **Knowledge and Understanding** Solve △ABC.



4. **Communication** How do you decide when to use the Sine Law or the Cosine Law to solve a triangle problem? Give examples to illustrate your explanation.

5. **Application** A car windshield wiper is 22 inches long. Through which angle did the blade in this diagram rotate?

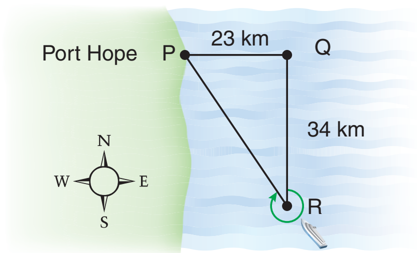


6. **Thinking** A sailboat leaves Port Hope and sails 23 km due east, then 34 km due south.

a) On what bearing will the boat travel on its way back to the starting point?

b) How far is the boat from the starting point?

c) What assumptions did you make to answer parts a and b?



Chapter Problem Designing a Stage

You have been asked to oversee the design and construction of a concert and theatre stage in your local community park.

1. Design a stage using at least two right triangles and at least two oblique triangles. Make decisions about the stage.
   - What will be the shape of the stage?
   - Will it have a roof?
   - Where will the stairs be? What will they look like? Will you include a ramp?
   - Include any further details you consider important for your stage.

2. Estimate reasonable angle measures and side lengths. Mark them on your drawing.

What tools did you use?

3. a) Describe two calculations for designing your stage that include a right triangle. Show your calculations.
   b) Describe two calculations for designing your stage that include an oblique triangle. Show your calculations.
   c) How could you use the results of your calculations from parts a and b? Justify your answer.

4. Tell what you like about your stage. Why is your stage appropriate for a concert or theatre? What would make someone choose your design for a stage?