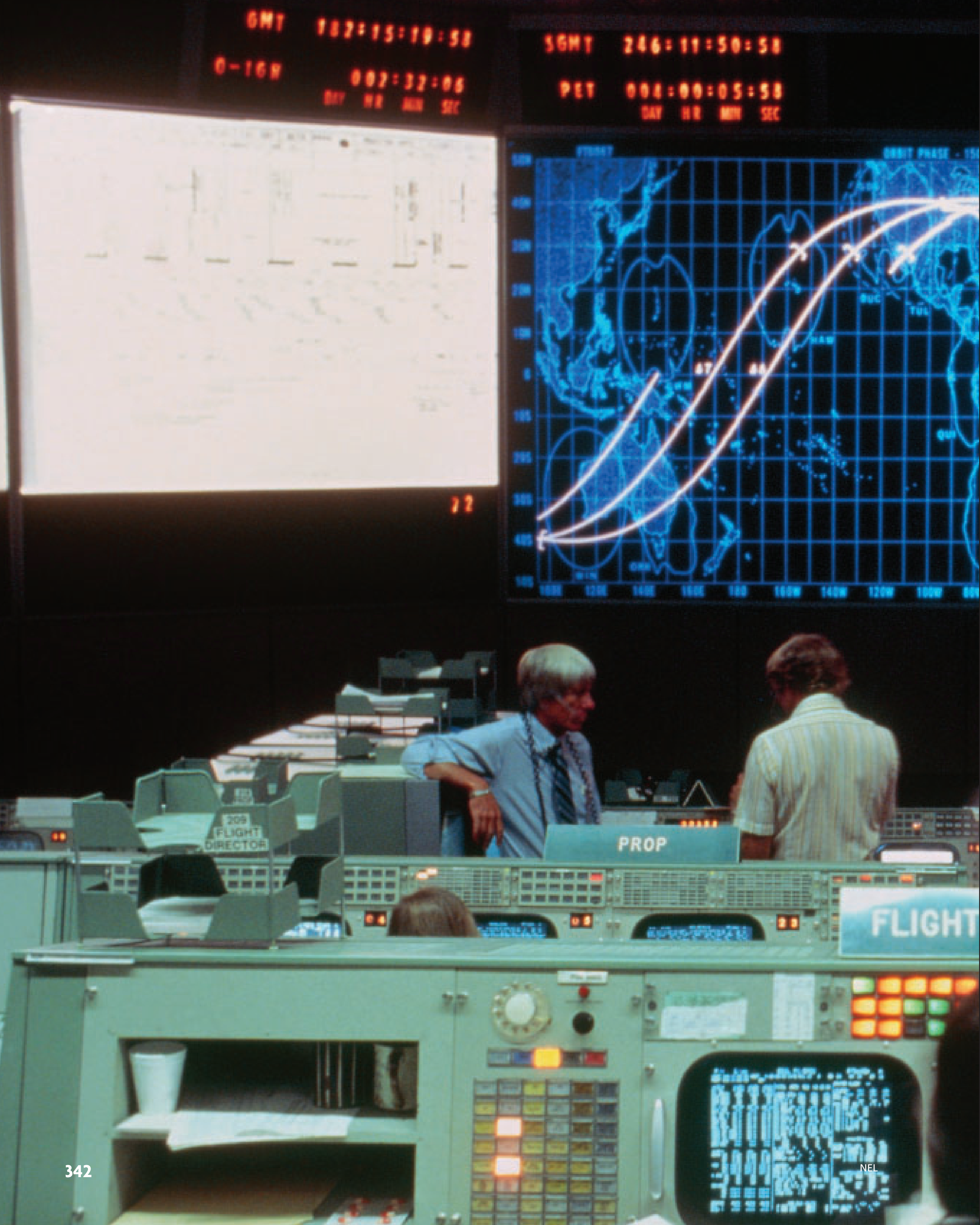
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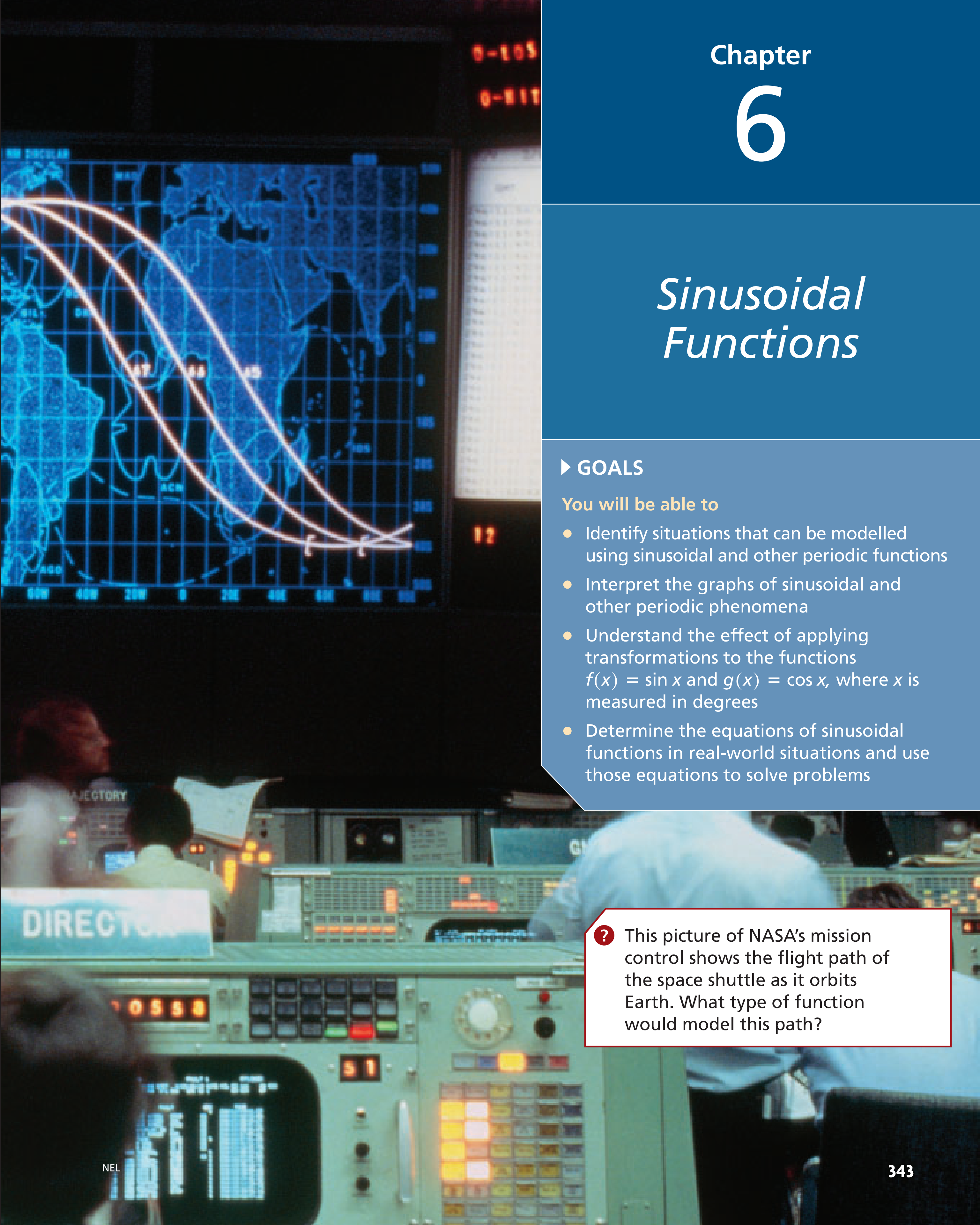


Chapter 6

Sinusoidal Functions

GOALS

You will be able to

- Identify situations that can be modelled using sinusoidal and other periodic functions
- Interpret the graphs of sinusoidal and other periodic phenomena
- Understand the effect of applying transformations to the functions $f(x) = \sin x$ and $g(x) = \cos x$, where x is measured in degrees
- Determine the equations of sinusoidal functions in real-world situations and use those equations to solve problems

? This picture of NASA's mission control shows the flight path of the space shuttle as it orbits Earth. What type of function would model this path?

6 Getting Started

Study Aid

- For help, see Essential Skills Appendix.

Question	Appendix
3, 4, 5	A-16
6, 7	A-14

SKILLS YOU NEED

1. Marcus sells 100 T-shirts per week at a price of \$30 per shirt. A survey indicates that if he reduces the price of each shirt by \$2, he will sell 20 more shirts per week. If x represents the number of times the price is reduced by \$2, then the revenue generated from T-shirt sales can be modelled by the function $R(x) = (30 - 2x)(100 + 20x)$.

a) Explain what the factors $(30 - 2x)$ and $(100 + 20x)$ represent in $R(x)$.

b) How many times will the price have to be dropped for the total revenue to be 0?

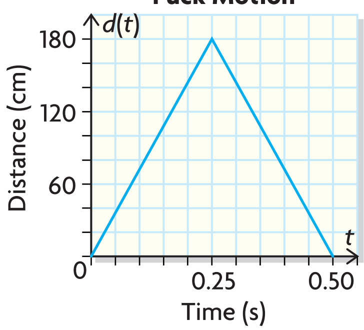
c) How many times will the price have to be dropped to reach the maximum revenue?

d) What is the maximum revenue?

e) What price will the T-shirts sell for to obtain the maximum revenue?

f) How many T-shirts will be sold to obtain the maximum revenue?

Puck Motion



2. An air hockey puck is shot to the opposite end of the table and ricochets back. The puck's distance in centimetres from where it was shot in terms of time in seconds can be modelled by the graph shown at the left.

a) How far did the puck travel?

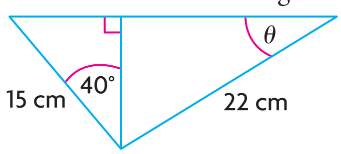
b) When was the puck farthest away from where it was shot?

c) How fast was the puck travelling in the first 0.25 s?

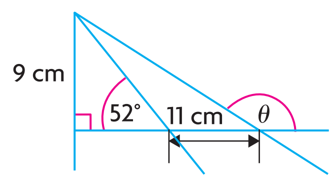
d) State the domain and range of the function.

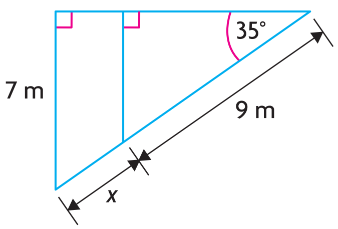
3. Determine θ to the nearest degree.

a)



b)





4. Determine the value of x in the triangle at the left to the nearest tenth of a metre.

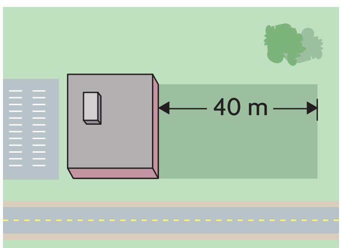
5. Use transformations of the graph $f(x) = 2^x$ to sketch the graphs of the following:

a) $y = -f(x)$

b) $y = 3f(x)$

c) $y = f(x) + 4$

d) $y = -2f(x - 3)$



6. An aerial photograph shows that a building casts a shadow 40 m long when the angle of elevation of the Sun was 32°. How tall is the building?

7. List all the different types of transformations that you know. For each one, describe how a graph of $f(x) = x^2$ would change if the transformation is applied to it.

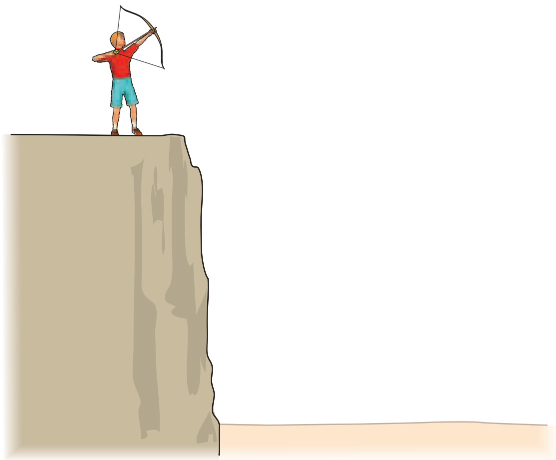
APPLYING What You Know

Flying Arrows

An arrow is shot into the air from the edge of a cliff. The height of the arrow above the ground is a function of time and can be modelled by

$$h(t) = -5t^2 + 20t + 25,$$

where the height, $h(t)$, is measured in metres at time, t, measured in seconds.



How can you describe the flight of the arrow using this function?

A. What is the initial height of the arrow?

B. Calculate $h(2)$. Explain what this value represents in this situation.

C. When will the arrow strike the ground?

D. When will the arrow reach its maximum height?

E. What is the maximum height reached by the arrow?

F. State the domain and range of the function in this situation.

G. Summarize what you determined about the relationship between the height of the arrow and time.

6.1 Periodic Functions and Their Properties

YOU WILL NEED

- graph paper

GOAL

Interpret and describe graphs that repeat at regular intervals.

LEARN ABOUT the Math

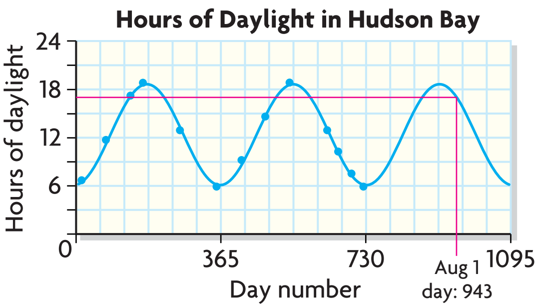
The number of hours of daylight at any particular location changes with the time of year. The table shows the average number of hours of daylight for approximately a two-year period at Hudson Bay, Nunavut. *Note*: Day 15 is January 15 of year 1. Day 74 is March 15 of year 1. Day 411 is February 15 of year 2.

Day	15	74	135	166	258	349	411	470	531	561	623	653	684	714
Hours of Daylight	6.7	11.7	17.2	18.8	12.9	5.9	9.2	14.6	18.8	18.1	12.9	10.2	7.5	5.9

? How many hours of daylight will there be on August 1 of year 3?

EXAMPLE 1 Representing data in a graph to make predictions

Jacob's Solution



I drew a scatter plot with the day as the independent variable and the hours of daylight as the dependent variable. I drew a smooth curve to connect the points.

The data and graph repeat every 365 days. I can tell because the greatest number of hours of daylight occurs on days 166 and 531, and $531 - 166 = 365$.

The least number of hours of daylight occurs on days 349 and 714, also 365 days apart.

I used the pattern to extend the graph to year 3. That would be 1095 days.

The number of hours of daylight for day 943 is about 17 h.

I used the graph to estimate the number of hours of daylight for day 943.

Reflecting

A. Why does it make sense to call the graph of the hours of daylight a periodic function?

B. How does the table help you predict the period of the graph?

C. Which points on the graph could you use to determine the range of this function?

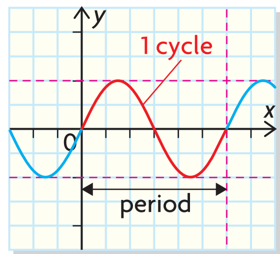
D. How does knowing the period of a periodic graph help you predict future events?

periodic function

a function whose graph repeats at regular intervals; the *y*-values in the table of values show a repetitive pattern when the *x*-values change by the same increment

period

the change in the independent variable (typically *x*) corresponding to one cycle; a cycle of a periodic function is a portion of the graph that repeats

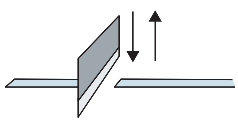


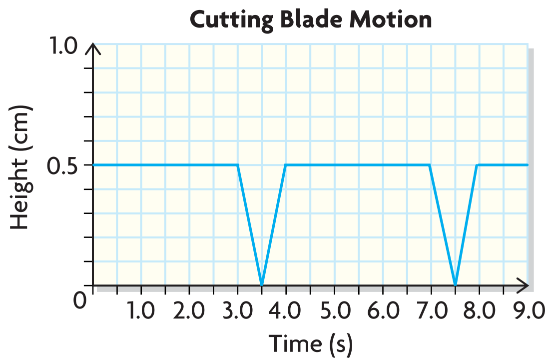
APPLY the Math

EXAMPLE 2 Interpreting periodic graphs and connecting them to real-world situations

Part A: Analyzing a Cutting Blade's Motion

Tanya's mother works in a factory that produces tape measures. One day, Tanya and her brother Norman accompany their mother to work. During manufacturing, a metal strip is cut into 6 m lengths and is coiled within the tape measure holder. A cutting machine chops the strips into their appropriate lengths. Tanya's mother shows a graph that models the motion of the cutting blade on the machine in terms of time.





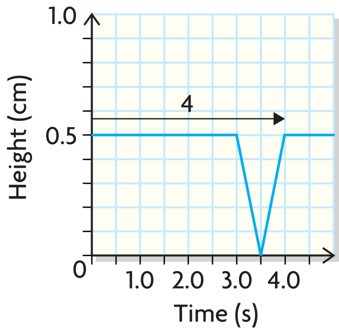
How can Norman interpret the graph and relate its characteristics to the manufacturing process?

Norman's Solution

This is a periodic function. ← It's a periodic function because the graph repeats in exactly the same way at regular intervals.

The period of this function is 4 s.

The cutting blade cuts a new section of metal strip every 4 s because the graph has a pattern that repeats every 4 s.



The maximum height of the blade is 0.5 cm. The minimum height is 0 cm.

The y-value is always 0.5 cm or less, so the blade can't be higher than this. When the height is 0 cm, the blade is hitting the cutting surface.

The blade stops for 3 s intervals.

Flat sections, like the ones from 0 to 3.0 and 4.0 to 7.0, must mean that the blade stops for these intervals. The machine is probably pulling the next 6 m section of metal strip through before it's cut.

The blade takes 1 s to go up and down.

Parts of the graph, like from $t = 3.0$ to $t = 3.5$, show that the blade takes 0.5 s to go down.

Other parts of the graph, like from $t = 3.5$ to $t = 4.0$, show that the blade takes 0.5 s to go up.

The blade will strike the cutting surface again at 11.5 s and every 4 s after that.

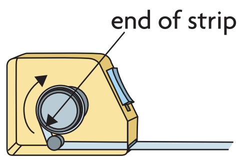
Since the graph repeats every 4 s and the blade hits the surface at 3.5 s and 7.5 s, I can figure out the next time it will hit the surface.

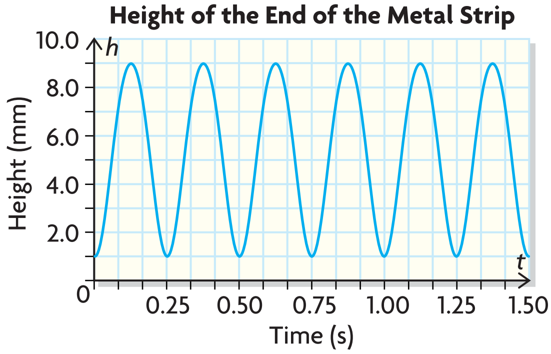
Part B: Analyzing the Motion of the Tape as It Is Spooled

Farther down the assembly line, the metal strip is raised and spooled onto a rotating cylinder contained within the tape measure.

Tanya notices that the height of the end of the metal strip that attaches to the spool goes up and down as the rest of the strip is pulled onto the cylinder.

Tanya's mother shows them a graph that models the height of the end of the strip in terms of time.





How can Tanya interpret the graph and relate its characteristics to the manufacturing process?

Tanya's Solution

This is a periodic function.

It's a periodic function because the graph repeats in exactly the same way at regular intervals. This time the action is smooth.

The range for this function is $\{h \in \mathbf{R} \mid 1 \leq h \leq 9\}$.

The highest the graph goes is 9 mm, and the lowest is 1 mm. The heights are always at or between these two values.

The period of this function is 0.25 s.

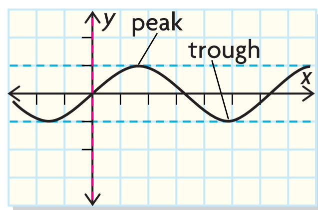
The first **trough** is at $t = 0$. The next trough is at $t = 0.25$. The distance between the two troughs gives the period.

I could also have measured the distance between the first two **peaks** to get that value.

The period represents the time it takes for the rotating cylinder to make one complete revolution.

trough
the minimum point on a graph

peak
the maximum point on a graph



equation of the axis

the equation of the horizontal line halfway between the maximum and the minimum; it is determined by

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

amplitude

half the difference between the maximum and minimum values; it is also the vertical distance from the function's axis to the maximum or minimum value

$$\frac{9 + 1}{2} = 5$$

I calculated the halfway point between the maximum and minimum values of the graph, giving me the **equation of the axis**.

The equation of the axis for this function is $h = 5$.

$$9 - 5 = 4$$

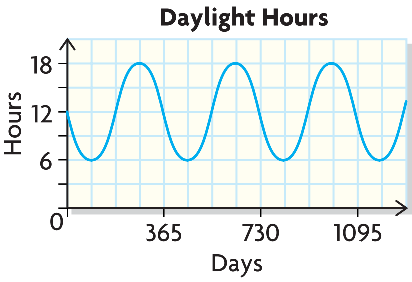
The **amplitude** of a function is the vertical distance from its axis $(h = 5)$ to its maximum value (9 mm).

The amplitude of this function is 4 mm.

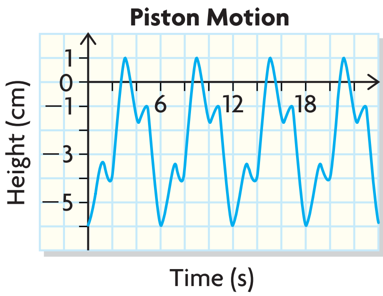
EXAMPLE 3 Identifying a periodic function from its graph

Determine whether the term *periodic* can be used to describe the graph for each situation. If so, state the period, equation of the axis, and amplitude.

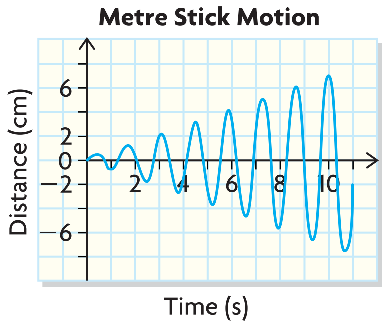
a) the average number of hours of daylight over a three-year period



b) the motion of a piston on an automated assembly line



c) a student is moving a metre stick back and forth with progressively larger movements



Tina's Solution

a) periodic

The graph looks like a series of waves that are the same size and shape. The waves repeat at regular intervals, so the function is periodic.

period = 1 year

The graph repeats its pattern every 365 days. That is the period of the function.

$$\frac{18 + 6}{2} = 12$$

equation of the axis: $h = 12$

To get the equation of the axis, I calculated the halfway point between the maximum and minimum values of the height.

$$18 - 12 = 6$$

amplitude = 6 h

The amplitude is the vertical distance from its axis $(h = 12)$ to the maximum value (18 h) or minimum value (6 h).

b) periodic

The shape of the graph repeats over the same interval, so the function is periodic.

period = 6 s

The graph repeats every 6 s, so that's the period of the function.

$$\frac{1 + (-6)}{2} = -2.5$$

The equation of the axis is halfway between the maximum of 1 and the minimum of −6.

equation of the axis: $h = -2.5$

$$1 - (-2.5) = 3.5$$

The distance between the maximum and the axis is 3.5.

amplitude = 3.5 cm

c) nonperiodic

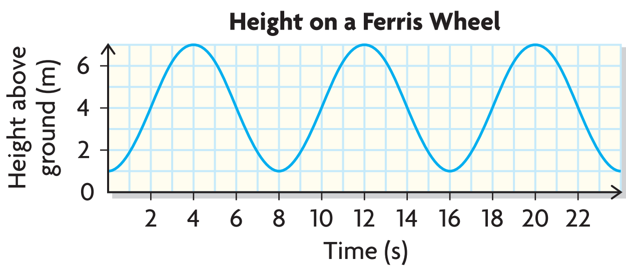
The shape of the graph does not repeat over the same interval, so the function is not periodic.

This means that the function does not have a period, amplitude, or equation of the axis.

In Summary

Key Ideas

- A function that produces a graph that has a regular repeating pattern over a constant interval is called a periodic function. It describes something that happens in a cycle, repeating in the same way over and over.



- A function that produces a graph that does not have a regular repeating pattern over a constant interval is called a nonperiodic function.

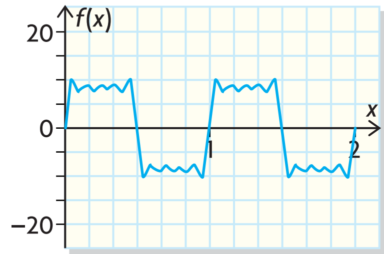
Need to Know

- Extending the graph of a periodic function by using the repeating pattern allows you to make reasonable predictions by extrapolating.
- The graph of a periodic function permits you to figure out the key features of the repeating pattern it represents, such as the period, amplitude, and equation of the axis.

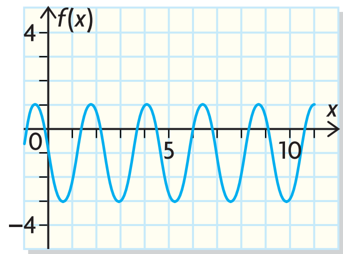
CHECK Your Understanding

1. Which of the following graphs are periodic? Explain why or why not.

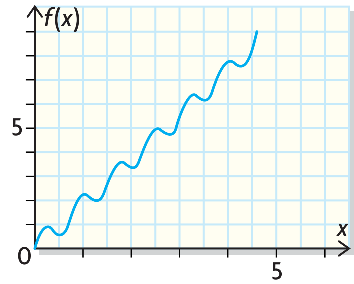
a)



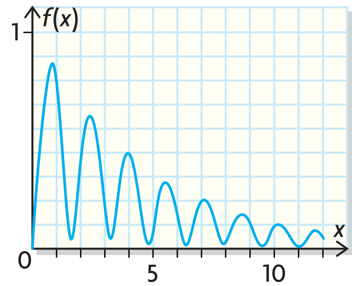
c)



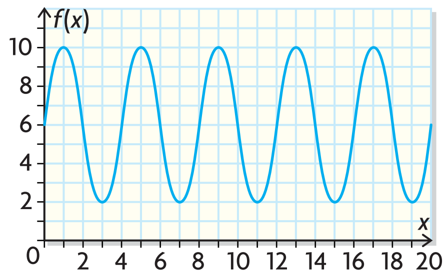
b)

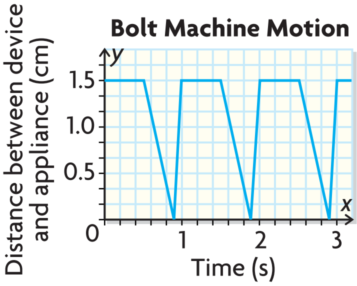


d)



2. Determine the range, period, equation of the axis, and amplitude of the function shown.





3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.

a) What is the period of one complete cycle?

b) What is the maximum distance between the device and the appliance?

c) What is the range of this function?

d) If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.

e) Determine the equation of the axis.

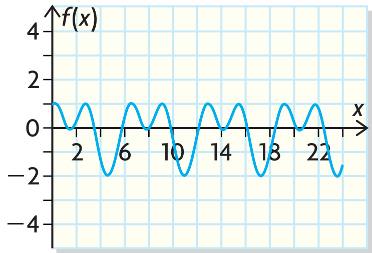
f) Determine the amplitude.

g) There are several parts to each complete cycle of the graph. Explain what each part could mean in the context of "attaching the bolt."

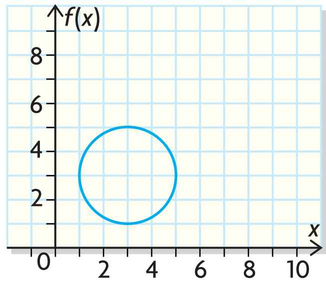
PRACTISING

4. Identify which graphs are periodic. Estimate the period of the functions that
K you identify as periodic.

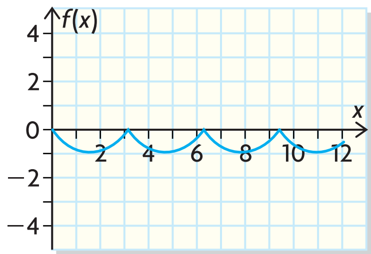
a)



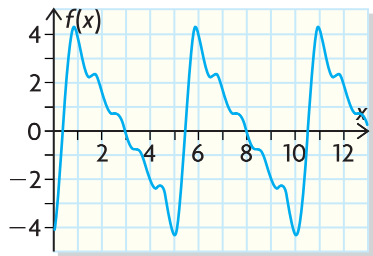
d)



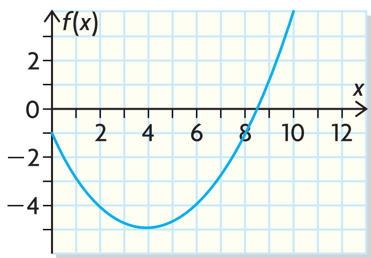
b)



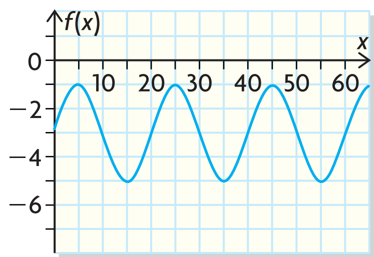
e)



c)



f)



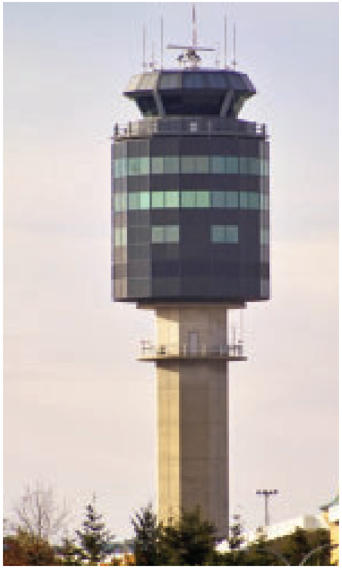
5. Which of the following situations would produce periodic graphs?

a) Sasha is monitoring the height of one of the cutting teeth on a chainsaw. The saw is on the ground, and the chain is spinning.
- independent variable: time
- dependent variable: height of tooth above the ground

b) Alex is doing jumping jacks.
- independent variable: time
- dependent variable: Alex's height above the ground

c) The cost of riding in a taxi varies, depending on how far you travel.
- independent variable: distance travelled
- dependent variable: cost

d) Brittany invested her money in a Guaranteed Investment Certificate whose return was 4% per year.
- independent variable: time
- dependent variable: value of the certificate



e) You throw a basketball to a friend, but she is so far away that the ball bounces on the ground four times.
- independent variable: distance
- dependent variable: bounce height

f) The antenna on a radar tower is rotating and emitting a signal to track incoming planes.
- independent variable: time
- dependent variable: intensity of the signal

6. Which of the tables of values might represent periodic functions? Justify.

a)

x	*y*
−5	9
−4	4
−3	1
−2	0
−1	1
0	4
1	9
2	16

b)

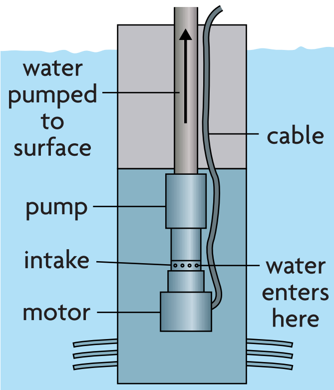
x	*y*
0.7	5
0.9	6
1.1	7
1.3	5
1.5	6
1.7	7
1.9	5
2.1	6

c)

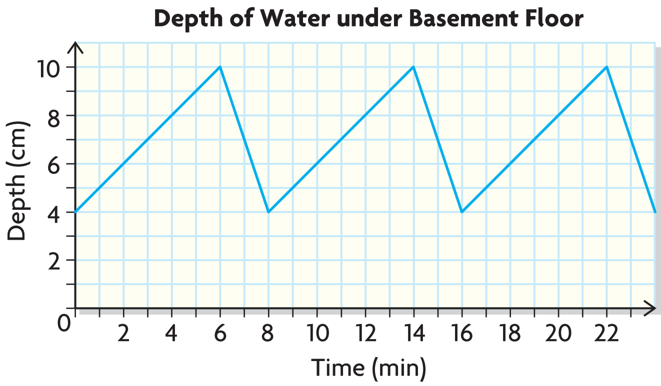
x	*y*
23	−6
26	−6.5
29	−7
32	−7.5
35	−8
38	−8.5
41	−9
44	−9.5

d)

x	*y*
1	5
2	6
4	5
7	6
11	5
16	6
22	5
29	6

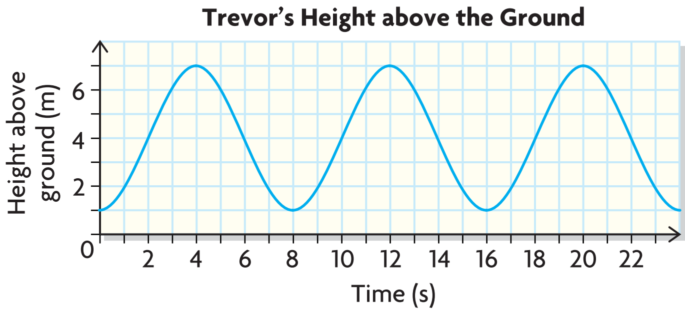


7. Chantelle has a submersible pump in her basement. During a heavy rain, the pump turned off and on to drain water collecting under her house's foundation. The graph models the depth of the water below her basement floor in terms of time. The depth of the water decreased when the pump was on and increased when the pump was off.



a) Is the function periodic?
b) At what depth does the pump turn on?
c) How long does the pump remain on?
d) What is the period of the function? Include the units of measure.
e) What is the range of the function?
f) What will the depth of the water be at 3 min?
g) When will the depth of the water be 10 cm?
h) What will the depth of the water be at 62 min?

8. While riding on a Ferris wheel, Trevor's height above the ground in terms of time can be represented by the graph shown.
A



a) What is the period of this function, and what does it represent?
b) What is the equation of the axis?
c) What is the amplitude?
d) What is the range of the function?
e) After 24 s, when will Trevor be at the lowest height again?
f) At what times is Trevor at the top of the wheel?
g) When will his height be 4 m between 24 s and 30 s?

9. Sketch the graph of a periodic function with a period of 20, an amplitude of 6, and whose equation of the axis is $y = 7$.

10. Sketch the graph of a periodic function whose period is 4 and whose range is $\{y \in \mathbf{R} \mid -2 \leq y \leq 5\}$.

11. Maria's bicycle wheel has a diameter of 64 cm. As she rides at a speed of 21.6 km/h, she picks up a stone in her tire. Draw a graph that shows the stone's height above the ground as she continues to ride at this speed for 2 s more.
T

12. A spacecraft is in an elliptical orbit around Earth. The spacecraft's distance above Earth's surface in terms of time is recorded in the table.

Time (min)	0	6	12	18	24	30	36	42	48	54	60	66	72	78
Distance (km)	550	869	1000	869	550	232	100	232	550	869	1000	869	550	232

a) Plot the data, and draw the resulting curve.
b) Is the graph periodic?
c) What is the period of the function, and what does it represent?
d) What is the approximate distance between the spacecraft and Earth at 8 min?
e) At what times is the spacecraft farthest from Earth?
f) If the spacecraft completes only six orbits before descending to Earth, what is the domain of the function?

13. Water is stored in a cylindrical container. Sometimes water is removed from the container, and other times water is added. The table records the depth of the water at specific times.

Time (min)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Depth (cm)	10	20	30	40	40	40	25	10	20	30	40	40	40	25	10	20	30	40	40	40

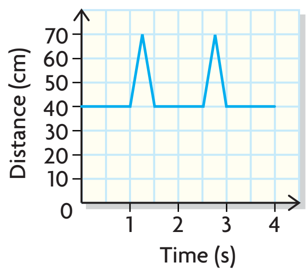
a) Plot the data, and draw the resulting curve.
b) Is the graph periodic?
c) Determine the period, the equation of the axis, and the amplitude of the function.
d) How fast is the depth of the water increasing when the container is being filled?
e) How fast is the depth of the water decreasing when the container is being drained?
f) Is the container ever empty? Explain.

14. **C** Write a definition of a periodic function. Include an example, and use your definition to explain why it is periodic.

Extension

15. A Calculator-Based Ranger (CBR) is a motion detector that can attach to a graphing calculator. When the CBR is activated, it records the distance an object is in front of the detector in terms of time. The data are stored in the calculator. A scatter plot based on those recorded distances and times can then be drawn using the graphing calculator. Distance is the dependent variable, and time is the independent variable. Denis holds the paddle of the CBR at 60 cm for 3 s and then, within 0.5 s, moves the paddle so that it is 30 cm from the detector. He holds the paddle there for 2 s and then, within 0.5 s, moves the paddle back to the 60 cm location. Denis repeats this process three times.
a) Draw a sketch of the resulting graph. Include a scale.
b) What is the period of the function?
c) Determine the range and domain of the function.

16. Describe the motion of the paddle in front of a CBR that would have produced the graph shown.



6.2 Investigating the Properties of Sinusoidal Functions

GOAL

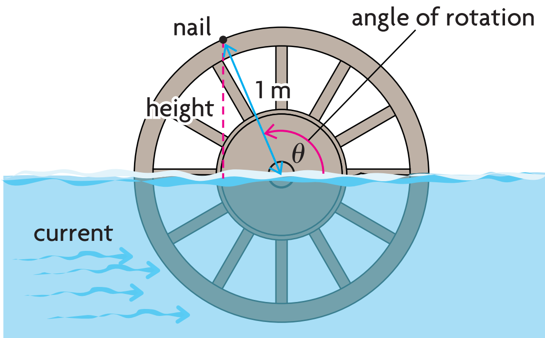
Examine the two functions that are associated with all sinusoidal functions.

YOU WILL NEED

- cardboard
- ruler
- protractor
- metre stick
- thumbtack
- graphing calculator

INVESTIGATE the Math

Paul uses a generator powered by a water wheel to produce electricity. Half the water wheel is submerged below the surface of a river. The wheel has a radius of 1 m. A nail on the circumference of the wheel starts at water level. As the current flows down the river, the wheel rotates counterclockwise to power the generator. The height of the nail changes as the wheel rotates.

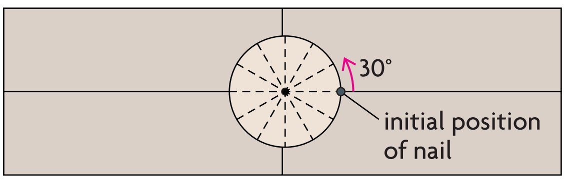


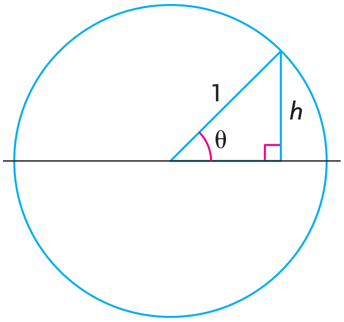
? How can you describe the position of the nail using an equation?

A. Construct a scale model of the water wheel. On a piece of cardboard, cut out a circle with a radius of 10 cm to represent the water wheel's 1 m radius.

B. Locate the centre of the circle. Use a protractor to divide your cardboard wheel into 30° increments through the centre. Draw a dot to represent the nail on the circumference of the circle at one of the lines you drew to divide the wheel.

C. On a rectangular piece of cardboard about 100 cm long and 30 cm wide, draw a horizontal line to represent the water level and a vertical line both through the centre. Attach the cardboard wheel to the centre of the rectangular piece of cardboard with a thumbtack, with the rectangle behind the wheel.



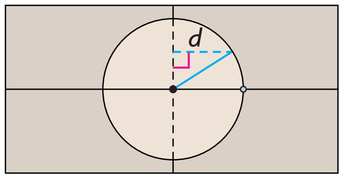


D. Rotate the cardboard wheel 30° counterclockwise. Measure the height, *h*, of the nail: the perpendicular distance from the nail to the horizontal line. Copy the table, and record the *actual* distance the nail is above the horizontal line at 30° by multiplying the scale height by 10 and converting to metres. Continue to rotate the wheel in 30° increments, measuring *h* and recording the actual heights. If the nail goes below the horizontal line, record the height as a negative value. Continue until the nail has rotated 720°.

Angle of Rotation, θ (°)	0	30	60	90	120	• • •	690	720
Actual Height of Nail, *h* (m)	0			1				

E. Use your data to graph height versus angle of rotation.

F. Use your model of the water wheel to examine the horizontal distance, *d*, the nail is from a vertical line that passes through the centre of the water wheel. Start with the nail initially positioned at water level.



Rotate the cardboard wheel 30° counterclockwise, and measure the distance the nail is from the vertical line. Copy the table, and record the *actual* distance the nail is from the vertical line at 30°, again adjusting for the scale factor. Continue to rotate the wheel in 30° increments, and record the actual distances. If the nail goes to the left of the vertical line, record the distance as a negative value. Continue until the nail has rotated 720°.

Angle of Rotation, θ (°)	0	30	60	90	120	• • •	690	720
Actual Distance from Vertical Line, *d* (m)	0			1				

G. Use your data to graph horizontal distance versus angle of rotation.

H. Use your graphing calculator to determine the cosine and sine of each rotation angle. Make sure your calculator is in DEGREE mode and evaluate to the nearest hundredth.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ													
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin θ													

Tech | *Support*

You can generate the tables using the List feature on your graphing calculator. Try putting degrees in L1, replacing L2 with "cos(L1)" and L3 with "sin(L1)."

I. Based on the tables you created in parts D, F, and H, select the appropriate equation that describes the height, h, of the nail on the water wheel in terms of the rotation. Also, identify another equation that describes the distance, d, the nail is from the vertical line in terms of the rotation.

$d(\theta) = \sin\theta \quad d(\theta) = 0.5\,\theta \quad \theta(d) = \sin d \quad d(\theta) = \cos\theta$

$h(\theta) = \sin\theta \quad h(\theta) = 0.5\,\theta \quad \theta(d) = \sin h \quad h(\theta) = \cos\theta$

Reflecting

J. Use your graphing calculator to graph $y = \sin x$ and $y = \cos x$, where $0° \leq x \leq 360°$, and compare these graphs to the graphs from parts E and G. Use words such as *amplitude*, *period*, *equation of the axis*, *increasing intervals*, *decreasing intervals*, *domain*, and *range* in your comparison.

K. State the coordinates of five key points that would allow you to draw the sinusoidal function $y = \sin x$ quickly over the interval $0°$ to $360°$.

L. State the coordinates of five key points that would allow you to draw the sinusoidal function $y = \cos x$ quickly over the interval $0°$ to $360°$.

M. What transformation can you apply to the cosine curve that will result in the sine curve?

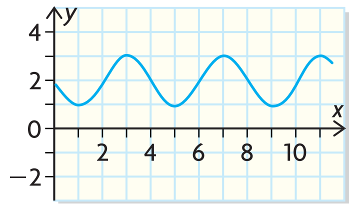
N. What ordered pair could you use to represent the point on the wheel that corresponds to the nail's location in terms of θ, the angle of rotation?

Tech Support

For help graphing trigonometric functions on your graphing calculator, see Technical Appendix, B-14.

sinusoidal function

a periodic function whose graph looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve; graphs of sinusoidal functions can be created by transforming the graph of the function $y = \sin x$ or $y = \cos x$

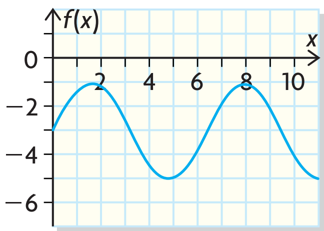


APPLY the Math

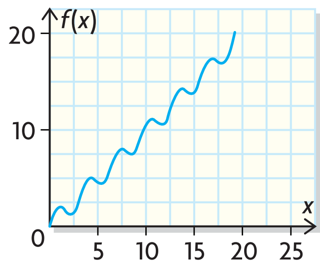
EXAMPLE 1 Identifying the function

Determine whether the graph represents a periodic function. If it does, determine whether it represents a sinusoidal function.

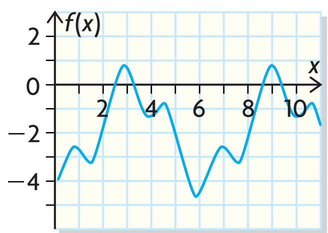
a)



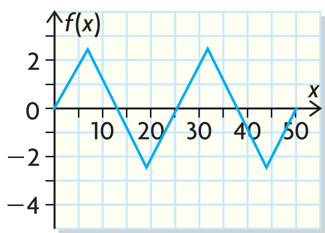
c)



b)



d)



Bridget's Solution

a) periodic and sinusoidal ← The function repeats, so it's periodic. It looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve.

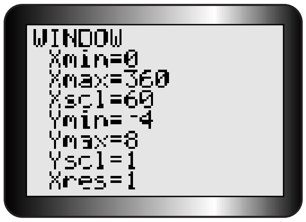
b) periodic ← The pattern repeats but the waves aren't symmetrical.

c) neither periodic nor sinusoidal ← It looks like smooth symmetrical waves; however, I can't horizontally translate any portion of the wave onto another portion of the curve.

d) periodic ← The pattern repeats but the waves aren't smooth curves.

EXAMPLE 2 Identifying the properties of a sinusoidal function

Graph the function $f(x) = 4 \sin(3x) + 2$ on a graphing calculator using the WINDOW settings shown in DEGREE mode.



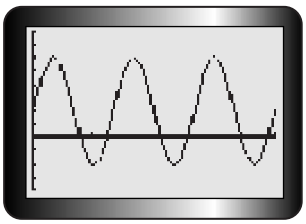
a) Is the function periodic? If it is, is it sinusoidal?

b) From the graph, determine the period, the equation of the axis, the amplitude, and the range.

c) Calculate $f(20°)$.

Beth's Solution

a)



← Because it repeats, the graph is periodic.

Since it forms a series of identical, symmetrical smooth waves, it is sinusoidal.

b) period $= 120°$

The graph completes three cycles in 360°, so one cycle, which is the period, must be 120°.

equation of the axis: $y = \frac{-2 + 6}{2}$

The axis is halfway between the minimum of −2 and the maximum of 6.

$y = 2$

$6 - 2 = 4$

amplitude $= 4$

To get the amplitude, I calculated the vertical distance between a maximum and the axis. It's 4.

range: $\{y \in \mathbf{R} \mid -2 \le y \le 6\}$

For the range, the greatest *y*-value on the graph (the maximum) is 6, and the least *y*-value (the minimum) is −2.

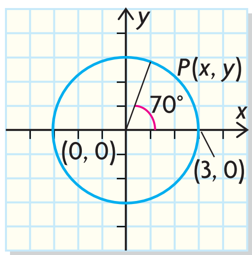
c)

$$f(x) = 4\sin(3x) + 2$$
$$f(20°) = 4\sin(3(20°)) + 2$$
$$= 4\sin(60°) + 2$$
$$\doteq 4(0.866) + 2$$
$$= 5.464$$

$f(20°)$ means find *y* when $x = 20°$. I substituted 20 for *x* and then calculated *y*.

EXAMPLE 3 Determining the coordinates of a point from a rotation angle

Determine the coordinates of the point $P(x, y)$ resulting from a rotation of 70° centred at the origin and starting from the point (3, 0).



Anne's Solution

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin\theta = \frac{y}{r}$$

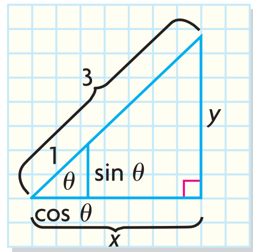
$$\frac{3}{1} = \frac{x}{\cos\theta} \quad \text{and} \quad \frac{3}{1} = \frac{y}{\sin\theta}$$

$$x = 3\cos\theta \qquad y = 3\sin\theta$$

$$P(x, y) = (3\cos\theta, 3\sin\theta)$$

The water wheel solution was based on a circle of radius 1. The coordinates of the nail after a rotation of θ were $(\cos\theta, \sin\theta)$. But this circle doesn't have a radius of 1. Its radius is 3.

I used similar triangles to figure out the coordinates of the larger triangle.



The coordinates for any point $P(x, y)$ on a circle of radius r are $P(x, y) = (r\cos\theta, r\sin\theta)$.

This means that the coordinates of the new point after a rotation of θ from the point $(r, 0)$ about $(0, 0)$ can be determined from $(r\cos\theta, r\sin\theta)$.

$$P(x, y) = (3\cos 70°, 3\sin 70°)$$
$$\doteq (1.03, 2.82)$$

I substituted the radius and angle of rotation into the ordered pair $(r\cos\theta, r\sin\theta)$ and got the coordinates of the image point.

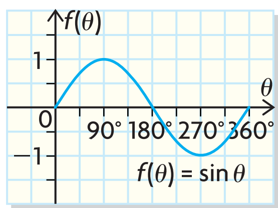
In Summary

Key Idea

- The function $f(\theta) = \sin\theta$ is a periodic function that represents the height (vertical distance) of a point from the x-axis as it rotates $\theta°$ about a circle with radius 1.
- The function $f(\theta) = \cos\theta$ is a periodic function that represents the horizontal distance of a point from the y-axis as it rotates $\theta°$ about a circle with radius 1.

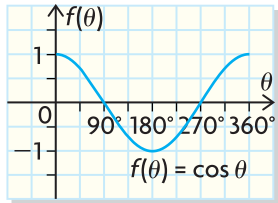
Need to Know

- The graph of $f(\theta) = \sin\theta$ has these characteristics:
  - The period is 360°.
  - The amplitude is 1, the maximum value is 1, and the minimum value is -1.
  - The domain is $\{\theta \in \mathbf{R}\}$, and the range is $-1 \le f(\theta) \le 1$.
  - The zeros are located at 0°, 180°, 360°, ...

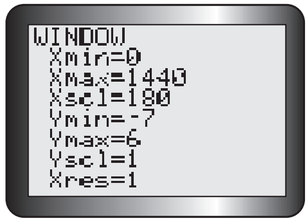


(continued)

- The graph of $f(\theta) = \cos\theta$ has these characteristics:
  - The period is 360°.
  - The amplitude is 1, the maximum value is 1, and the minimum value is -1.
  - The domain is $\{\theta \in \mathbf{R}\}$, and the range is $-1 \leq f(\theta) \leq 1$.
  - The zeros are located at 90°, 270°, 450°, ...
- The sine function and cosine function are congruent sinusoidal curves; the cosine curve is the sine curve translated 90° to the left.
- Any point $P(x, y)$ on a circle centred at (0, 0) with radius r and rotated through an angle θ can be expressed as an ordered pair $(r\cos\theta, r\sin\theta)$.



CHECK Your Understanding



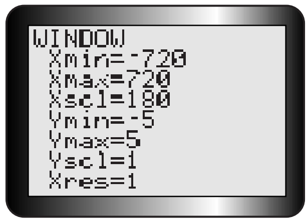
1. Using a graphing calculator in DEGREE mode, graph each sinusoidal function. Use the WINDOW settings shown. From the graph, state the amplitude, period, and equation of the axis for each.
a) $y = 3\sin(2x) + 1$ **b)** $y = 4\cos(0.5x) - 2$

2. a) If $h(x) = \sin(5x) - 1$, calculate $h(25°)$.
b) If $f(x) = \cos x$ and $f(x) = 0$, list the values of x where $0° \leq x \leq 360°$.

3. A buoy rises and falls as it rides the waves. The equation $h(t) = \cos(36t)°$ models the displacement of the buoy, $h(t)$, in metres at t seconds.
a) Graph the displacement from 0 s to 20 s, in 2.5 s intervals.
b) Determine the period of the function from the graph.
c) What is the displacement at 35 s?
d) At what time, to the nearest second, does the displacement first reach -0.8 m?

4. Determine the coordinates of the new point after a rotation of 50° about (0, 0) from the point (2, 0).

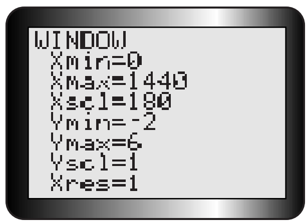
PRACTISING



5. Using a graphing calculator and the WINDOW settings shown, graph each function. Use DEGREE mode. State whether the resulting functions are periodic. If so, state whether they are sinusoidal.
a) $y = 3\sin x + 1$ **c)** $y = \cos(2x) - \sin x$ **e)** $y = 0.5\cos x - 1$
b) $y = (0.004x)\sin x$ **d)** $y = 0.005x + \sin x$ **f)** $y = \sin 90°$

6. Based on your observations in question 5, what can you conclude about any function that possesses sine or cosine in its equation?

7. If $g(x) = \sin x$ and $h(x) = \cos x$, where $0° \leq x \leq 360°$, calculate each and explain what it means.
a) $g(90°)$ **b)** $h(90°)$



8. **K** Using a graphing calculator in DEGREE mode, graph each sinusoidal function. Use the WINDOW settings shown. From the graph, state the amplitude, period, increasing intervals, decreasing intervals, and equation of the axis for each.
a) $y = 2\sin x + 3$ **c)** $y = \sin(0.5x) + 2$ **e)** $y = 2\sin(0.25x)$
b) $y = 3\sin x + 1$ **d)** $y = \sin(2x) - 1$ **f)** $y = 3\sin(0.5x) + 2$

9. **a)** If $f(x) = \cos x$, calculate $f(35°)$.
b) If $g(x) = \sin(2x)$, calculate $g(10°)$.
c) If $h(x) = \cos(3x) + 1$, calculate $h(20°)$.
d) If $f(x) = \cos x$ and $f(x) = -1$, calculate x for $0° \leq x \leq 360°$.
e) If $f(x) = \sin x$ and $f(x) = -1$, calculate x for $0° \leq x \leq 360°$.

10. Determine all values where $\sin x = \cos x$ for $-360° \leq x \leq 360°$.

T
11. **a)** Determine the coordinates of the new point after a rotation of 25° about (0, 0) from the point (1, 0).
b) Determine the coordinates of the new point after a rotation of 80° about (0, 0) from the point (5, 0).
c) Determine the coordinates of the new point after a rotation of 120° about (0, 0) from the point (4, 0).
d) Determine the coordinates of the new point after a rotation of 230° about (0, 0) from the point (3, 0).

12. Sketch the sinusoidal graphs that satisfy the properties in the table.

	Period	Amplitude	Equation of the Axis	Number of Cycles
a)	4	3	$y = 5$	2
b)	20	6	$y = 4$	3
c)	80	5	$y = -2$	2

13. Jim is riding a Ferris wheel, where t is time in seconds. Explain what each of
A the following represents.
a) $h(10)$, where $h(t) = 5\cos(18t)°$
b) $h(10)$, where $h(t) = 5\sin(18t)°$

14. Compare the graphs for $y = \sin x$ and $y = \cos x$, where $0° \leq x \leq 360°$.
C How are they the same, and how are they different?

Extending

15. If the water level in the original water wheel situation was lowered so that three-quarters of the wheel was exposed, determine the equation of the sinusoidal function that describes the height of the nail in terms of the rotation.

16. A spring bounces up and down according to the model $d(t) = 0.5\cos(120t)°$, where $d(t)$ is the displacement in centimetres from the rest position and t is time in seconds. The model does not consider the effects of gravity.
a) Make a table for $0 \leq t \leq 9$. Use 0.5 s intervals.
b) Draw the graph.
c) Explain why the function models periodic behaviour.
d) What is the relationship between the amplitude of the function and the displacement of the spring from its rest position?

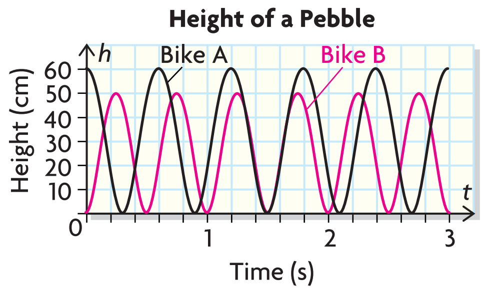
6.3 Interpreting Sinusoidal Functions

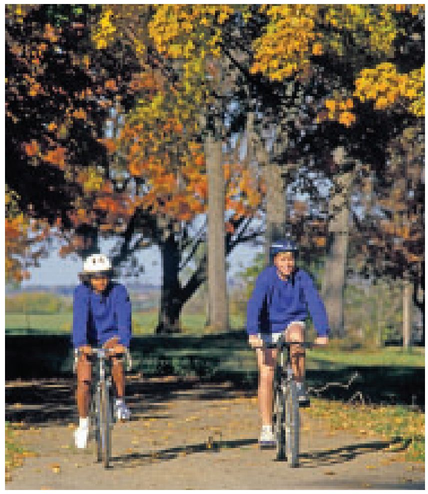
GOAL

Relate details of sinusoidal phenomena to their graphs.

LEARN ABOUT the Math

Two students are riding their bikes. A pebble is stuck in the tire of each bike. The two graphs show the heights of the pebbles above the ground in terms of time.





? What information about the bikes can you gather from the graphs of these functions?

EXAMPLE 1 Connecting the graph of a sinusoidal function to the situation

Joanne's Solution: Comparing Peaks of a Sinusoidal Function

For Bike A, the pebble was initially at its highest height of 60 cm. For Bike B, the pebble was initially at its lowest height of 0 cm.

For Bike A, the graph starts at a peak. For Bike B, the graph starts at a trough.

The wheels have different diameters. The diameter of the wheel on Bike A is 60 cm. The diameter of the wheel on Bike B is 50 cm.

I noticed that the peaks on the graph are different. The peak for Bike A is at $h = 60$, which is greater than the peak for Bike B, which is at $h = 50$. The troughs, however, are the same, $h = 0$.

Glen's Solution: Comparing Periods

The wheel on Bike A takes 0.6 s to complete one revolution. The wheel on Bike B takes 0.5 s to complete one revolution.

The period of Bike A is 0.6 s. The period of Bike B is 0.5 s.

The graph for Bike A completes 5 cycles in 3 s, so the period, or length of one cycle, is 0.6 s.

The graph for Bike B completes 2 cycles in 1 s, so the period is 0.5 s.

Scott's Solution: Comparing Equations of the Axes in Sinusoidal Functions

Bike A: $\frac{60 + 0}{2} = 30$

Bike B: $\frac{50 + 0}{2} = 25$

The axis is halfway between a peak (or maximum) and a trough (or minimum). I added the maximum and the minimum and then divided by 2.

The equation of the axis for Bike A is $h = 30$.

The equation of the axis for Bike B is $h = 25$.

The axle for the wheel on Bike A is 30 cm above the ground. The axle for the wheel on Bike B is 25 cm above the ground.

Karen's Solution: Comparing Speeds

Circumference:

Speed is equal to distance divided by time, so first I had to figure out how far each bike travels when the wheel completes one revolution. This distance is the circumference. I calculated the two circumferences.

Bike A	Bike B
$C_A = 2\pi r_A$	$C_B = 2\pi r_B$
$C_A = 2\pi(30)$	$C_B = 2\pi(25)$
$C_A = 60\pi$	$C_B = 50\pi$
$C_A \doteq 188.5$ cm	$C_B \doteq 157.1$ cm
$C_A \doteq 1.885$ m	$C_B \doteq 1.571$ m
$s_A = \frac{d}{t}$	$s_B = \frac{d}{t}$
$s_A = \frac{1.885}{0.6}$	$s_B = \frac{1.571}{0.5}$
$s_A \doteq 3.14$ m/s	$s_B \doteq 3.14$ m/s

To calculate the speed, I divided each circumference by the time taken to complete one revolution.

The bikes are travelling at the same speed.

Reflecting

A. How would changing the speed of the bike affect the sinusoidal graph?

B. For a third rider travelling at the same speed but on a bike with a larger wheel than that on Bike A, how would the graph of the resulting sinusoidal function compare with that for Bike A and Bike B?

C. What type of information can you learn by examining the graph modelling the height of a pebble stuck on a tire in terms of time?

APPLY the Math

EXAMPLE 2 Comparing graphs and situations

Annette's shop teacher was discussing table saws. The teacher produced two different graphs for two different types of saw. In each case, the graphs show the height of one tooth on the circular blade relative to the cutting surface of the saw in terms of time.

Table Saw A

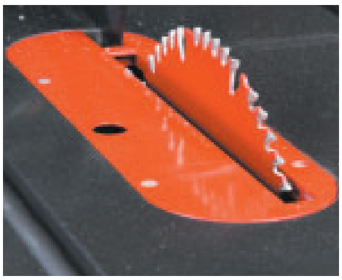


Table Saw B

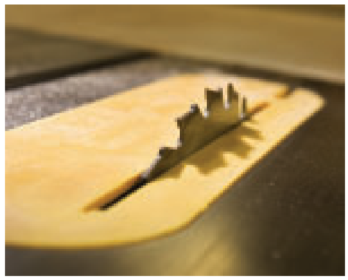


Table Saw A

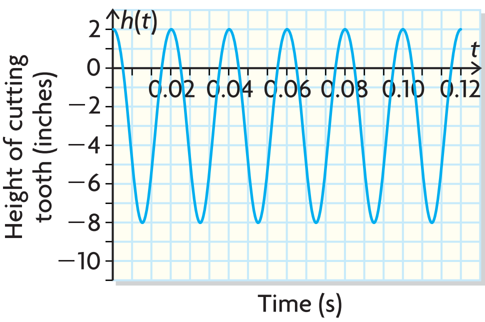
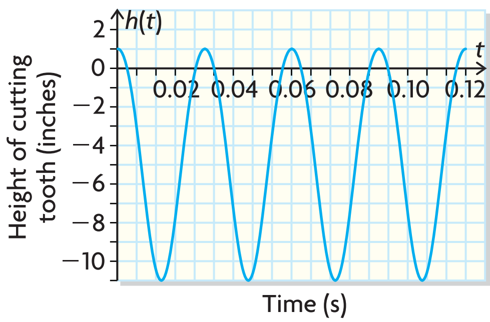


Table Saw B



What information about the table saws can Annette gather from the graphs?

Repko's Solution

The blade on Table Saw A is set higher than the blade on Table Saw B.

The peaks on the graph are different. The peak for A is at $h = 2$; the peak for B is at $h = 1$.

The blade on Table Saw A takes 0.02 s to complete one revolution.

One of the easiest ways to find the period is to figure out how long it takes to go from one peak on the graph to the next.

On graph A, the first peak is at 0 s, and the next is at 0.02 s. This means that the period of graph A is 0.02 s.

The blade on Table Saw B takes 0.03 s to complete one revolution.

On graph B, the first peak is at 0 s, and the next is at 0.03 s. The period of graph B is 0.03 s.

The axle for the blade on Table Saw A is 3 in. below the cutting surface.

For graph A, I found the equation of the axis by adding 2 and -8 and then dividing by 2. That gave me -3. The equation of the axis for graph A is $h = -3$.

The axle for the blade on Table Saw B is 5 in. below the cutting surface.

For graph B, I added 1 and -11 and then divided by 2. That gave me -5. The equation of the axis for graph B is $h = -5$.

The radius of the circular cutting blade on Table Saw A is 5 in.

The radius of the circular cutting blade on Table Saw B is 6 in.

For graph A, I got the amplitude by taking the difference between 2 and -3. The amplitude for graph A is 5.

For graph B, the amplitude is the difference between 1 and -5. The amplitude for graph B is 6.

In both cases, the distance from the axis to a peak represents the radius of the circular cutting blade.

EXAMPLE 3 Using technology to understand a situation

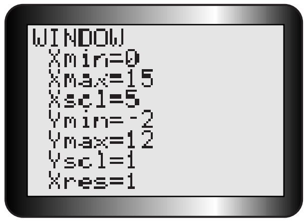
The function $j(t) = 4.1 \sin(64.7t)^\circ + 5.8$, where t is time in years since May 1992 and $j(t)$ is the number of applications for jobs each week (in hundreds), models demand for employment in a particular city.

a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph the function and then sketch the graph.

b) How long is the employment cycle? Explain how you know.

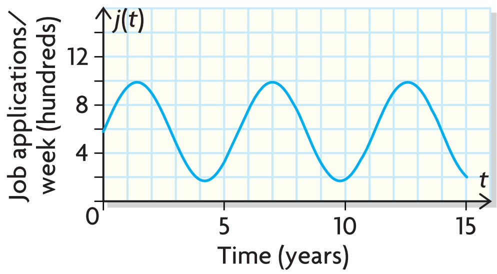
c) What is the minimum number of applications per week in this city?

d) Calculate $j(10)$, and explain what it represents in terms of the situation.



Karl's Solution

a)



I sketched all the cycles the window showed.

b) $6.9 - 1.34 = 5.56$

The employment cycle is 5.56 years, the distance between peaks or troughs.

To calculate the cycle, I calculated the *x*-interval between the first and second peak.

c) The minimum number of applications per week is 170.

I looked for a trough on the graph and read the *j*-coordinate.

d) $j(10) = 1.88$

There were 188 applications in May 2002.

I looked for the place where the *t*-coordinate was 10.

In Summary

Key Idea

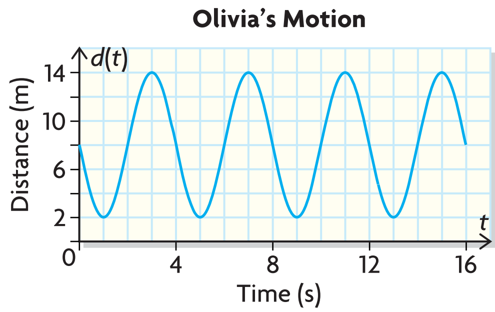
- The sine and cosine functions can be used as models to solve problems that involve many types of repetitive motions and trends.

Need to Know

- If a situation can be described by a sinusoidal function, the graph of the data should form a series of symmetrical waves that repeat at regular intervals. The amplitude of the sine or cosine function depends on the situation being modelled.
- One cycle of motion corresponds to one period of the sine function.
- The distance of a circular path is calculated from the circumference of the path. The speed of an object following a circular path can be calculated by dividing the distance by the period, the time to complete one rotation.

CHECK *Your Understanding*

1. Olivia was swinging back and forth in front of a motion detector when the detector was activated. Her distance from the detector in terms of time can be modelled by the graph shown.



a) What is the equation of the axis, and what does it represent in this situation?

b) What is the amplitude of this function?

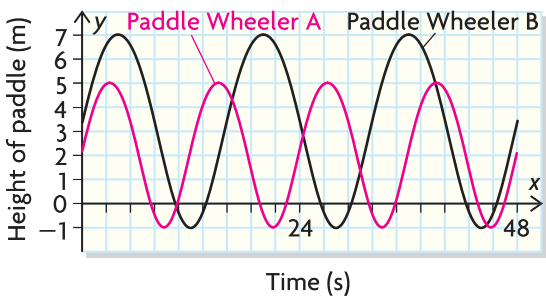
c) What is the period of this function, and what does it represent in this situation?

d) How close did Olivia get to the motion detector?

e) At $t = 7$ s, would it be safe to run between Olivia and the motion detector? Explain your reasoning.

f) If the motion detector was activated as soon as Olivia started to swing from at rest, how would the graph change? (You may draw a diagram or a sketch.) Would the resulting graph be sinusoidal? Why or why not?

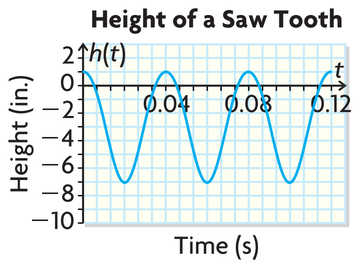
2. Marianna collected some data on two paddle wheels on two different boats and constructed two graphs. Analyze the graphs, and explain how the wheels differ. Refer to the radius of each wheel, the height of the axle relative to the water, the time taken to complete one revolution, and the speed of each wheel.



3. Draw two sinusoidal functions that have the same period and axes but have different amplitudes.

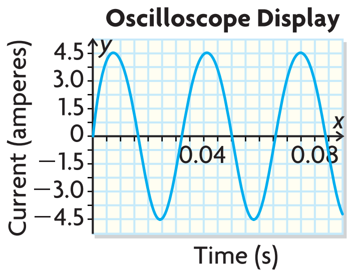
PRACTISING

4. Evan's teacher gave him a graph to help him understand the speed at which a tooth on a saw blade travels. The graph shows the height of one tooth on the circular blade relative to the cutting surface relative to time.
K



a) How high above the cutting surface is the blade set?
b) What is the period of the function, and what does it represent in this situation?
c) What is the amplitude of the function, and what does it represent in this situation?
d) How fast is a tooth on the circular cutting blade travelling in inches per second?

5. An oscilloscope hooked up to an alternating current (AC) circuit shows a sine curve on its display.



a) What is the period of the function? Include the units of measure.
b) What is the equation of the axis of the function? Include the units of measure.
c) What is the amplitude of the function? Include the units of measure.

6. Sketch a height-versus-time graph of the sinusoidal function that models each situation. Draw at least three cycles. Assume that the first point plotted on each graph is at the lowest possible height.
a) A Ferris wheel with a radius of 7 m, whose axle is 8 m above the ground, and that rotates once every 40 s
b) A water wheel with a radius of 3 m, whose centre is at water level, and that rotates once every 15 s
c) A bicycle tire with a radius of 40 cm and that rotates once every 2 s
d) A girl lying on an air mattress in a wave pool that is 3 m deep, with waves 0.5 m in height that occur at 7 s intervals

7. The tables show the varying length of daylight for Timmins, Ontario, located at a latitude of 48°, and Miami, Florida, located at a latitude of 25°. The length of the day is calculated as the interval between sunrise and sunset.

Timmins, at latitude 48°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	8.8	10.2	11.9	13.7	15.2	16.1	15.7	14.4	12.6	10.9	9.2	8.3

Miami, at latitude 25°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	10.7	11.3	12.0	12.8	13.6	13.8	13.6	13.1	12.3	11.6	10.9	10.5

a) Plot the data on separate coordinate systems, and draw a smooth curve through each set of points.

b) Compare the two curves. Refer to the periods, amplitudes, and equations of the axes.

c) What might you infer about the relationship between hours of daylight and the latitude at which you live?

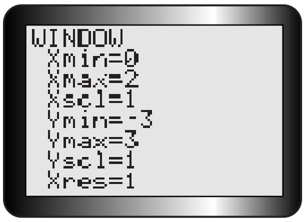
8. **T** The diameter of a car's tire is 52 cm. While the car is being driven, the tire picks up a nail.

a) Draw a graph of the height of the nail above the ground in terms of the distance the car has travelled since the tire picked up the nail.

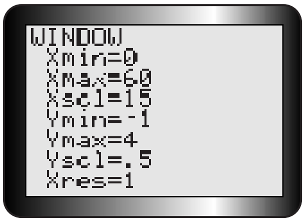
b) How high above the ground will the nail be after the car has travelled 0.1 km?

c) How far will the car have travelled when the nail reaches a height of 20 cm above the ground for the fifth time?

d) What assumption must you make concerning the driver's habits for the function to give an accurate height?



9. In high winds, the top of a signpost vibrates back and forth. The distance the tip of the post vibrates to the left and right of its resting position can be defined by the function $d(t) = 3\sin(1080t)°$, where $d(t)$ represents the distance in centimetres at time t seconds. If the wind speed decreases by 20 km/h, the vibration of the tip can be modelled by the function $d(t) = 2\sin(1080t)°$. Using graphing technology in DEGREE mode and the WINDOW settings shown, produce the two graphs. How does the reduced wind speed affect the period, amplitude, and equation of the axis?



10. The height, $h(t)$, of a basket on a water wheel at time t can be modelled by $h(t) = 2\sin(12t) + 1.5°$, where t is in seconds and $h(t)$ is in metres.

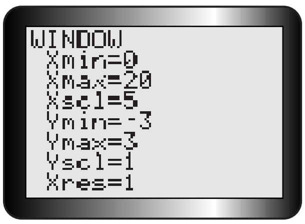
a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $h(t)$ and sketch the graph.

b) How long does it take for the wheel to make a complete revolution? Explain how you know.

c) What is the radius of the wheel? Explain how you know.

d) Where is the centre of the wheel located in terms of the water level? Explain how you know.

e) Calculate $h(10)$, and explain what it represents in terms of the situation.



11. **A** The equation $h(t) = 2.5\sin(72t)°$ models the displacement of a buoy in metres at t seconds.

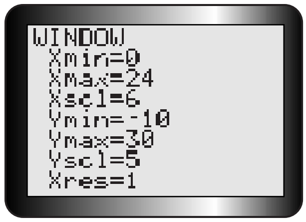
a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $h(t)$ and sketch the graph.

b) How long does it take for the buoy to travel from the peak of a wave to the next peak? Explain how you know.

c) How many waves will cause the buoy to rise and fall in 1 min? Explain how you know.

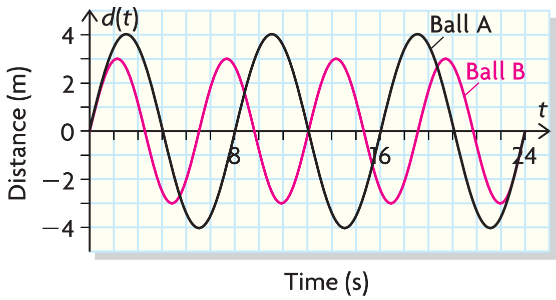
d) How far does the buoy drop from its highest point to its lowest point? Explain how you know.

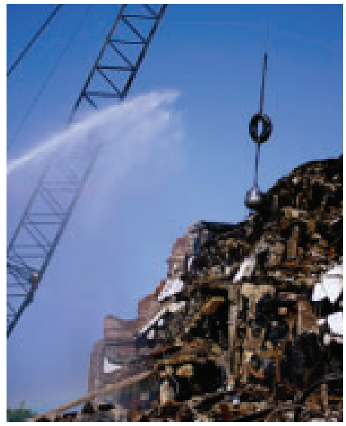
12. The average monthly temperature, $T(t)$, in degrees Celsius in Kingston, Ontario, can be modelled by the function $T(t) = 14.2 \sin(30(t - 4.2))° + 5.9$, where t represents the number of months. For $t = 1$, the month is January; for $t = 2$, the month is February; and so on.



a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $T(t)$ and sketch the graph.
b) What does the period represent in this situation?
c) What is the average temperature range in Kingston?
d) What is the mean temperature in Kingston?
e) Calculate $T(30)$, and explain what it represents in terms of the situation.

13. Two wrecking balls attached to different cranes swing back and forth. The distance the balls move to the left and the right of their resting positions in terms of time can be modelled by the graphs shown.



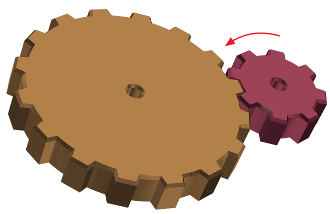


a) What is the period of each function, and what does it represent in this situation?
b) What is the equation of the axis of each function, and what does it represent in this situation?
c) What is the amplitude of each function, and what does it represent in this situation?
d) Determine the range of each function.
e) Compare the motions of the two wrecking balls.

14. How many pieces of information do you need to know to sketch a sinusoidal function. What pieces of information could they be?
C

Extending

15. A gear of radius 1 m turns counterclockwise and drives a larger gear of radius 4 m. Both gears have their axes along the horizontal.



a) In which direction is the larger gear turning?
b) If the period of the smaller gear is 2 s, what is the period of the larger gear?
c) In a table, record convenient intervals for each gear, to show the vertical displacement, d, of the point where the two gears first touched. Begin the table at 0 s and end it at 24 s. Graph vertical displacement versus time.
d) What is the displacement of the point on the large wheel when the drive wheel first has a displacement of -0.5 m?
e) What is the displacement of the drive wheel when the large wheel first has a displacement of 2 m?
f) What is the displacement of the point on the large wheel at 5 min?

6 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 6.2, Examples 1 and 2.
- Try Mid-Chapter Review Question 3.

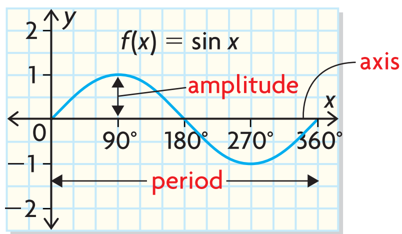
Q: What are sinusoidal functions, and what characteristics are often used to describe them?

A: Sinusoidal functions, like other periodic functions, repeat at regular intervals. Unlike other periodic functions, sinusoidal functions form smooth symmetrical waves such that any portion of a wave can be horizontally translated onto another portion of the curve. Sinusoidal functions are formed from transformations of the functions $y = \sin x$ and $y = \cos x$.

The three characteristics of a sinusoidal function, as well as any periodic function, are the period, the equation of the axis, and the amplitude.

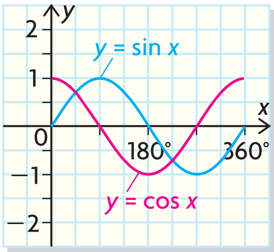
Period	Equation of the Axis	Amplitude
The period is the change in x corresponding to one cycle. (A cycle of a sinusoidal function is a portion of the graph that repeats.) One way to determine the period is to look at the change in x between two maxima.	The equation of the axis is the equation of the line halfway between the maximum and minimum values on a sinusoidal function. It can be determined with the formula $y = \frac{(\text{maximum value} + \text{minimum value})}{2}$	The amplitude is the vertical distance from the function's axis to the minimum or maximum value. It is always positive.

EXAMPLE



For the function $f(x) = \sin x$, the period is 360°, the equation of the axis is $y = 0$, and the amplitude is 1.

Q: How do the graphs of $y = \sin x$ and $y = \cos x$ compare?



A: Similarities

The period is 360°.

The equation of the axis is $y = 0$.

The amplitude $= 1$.

The range is $\{y \in \mathbf{R} \mid -1 \le y \le 1\}$.

Differences

A maximum for $y = \sin x$ occurs at 90° and at increments of 360° from that point.

A maximum for $y = \cos x$ occurs at 0° and at increments of 360° from that point.

A minimum for $y = \sin x$ occurs at 270° and at increments of 360° from that point.

A minimum for $y = \cos x$ occurs at 180° and at increments of 360° from that point.

The graph of the function $y = \sin x$ can be changed to a graph of the function $y = \cos x$ by applying a horizontal translation of 90° to the left.

The graph of the function $y = \cos x$ can be changed to a graph of the function $y = \sin x$ by applying a horizontal translation of 90° to the right.

Q: Why might it be useful to learn about sinusoidal functions?

A: Many real-world phenomena that have a regular repeating pattern can be modelled with sinusoidal functions. For example,

- the motion of objects in a circular orbit
- the motion of swinging objects, such as a pendulum
- the number of hours of sunlight for a particular latitude
- the phase of the Moon
- the current for an AC circuit

Study Aid

- See Lesson 6.3, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 5 and 6.

PRACTICE Questions

Lesson 6.1

1. Sketch the graph of a periodic function whose period is 10 and whose range is $\{y \in \mathbf{R} \mid 4 \leq y \leq 10\}$.

2. The following data show the pressure (in pounds per square inch, psi) in the tank of an air compressor at different times.

Time (s)	0	1	2	3	4	5	6	7	8	9
Pressure (psi)	60	60	80	100	100	90	80	70	60	60

Time (s)	10	11	12	13	14	15	16	17	18	19
Pressure (psi)	80	100	100	90	80	70	60	60	80	100

a) Create a scatter plot of the data and the curve that best models the data.
b) How do you know that the graph is periodic?
c) Determine the period of the function.
d) Determine the equation of the axis.
e) Determine the amplitude.
f) How fast is the air pressure increasing when the compressor is on?
g) How fast is the air pressure decreasing when the equipment is in operation?
h) Is the container ever empty? Explain.

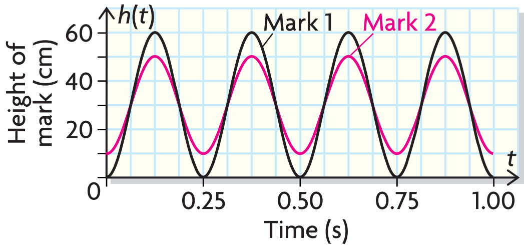
Lesson 6.2

3. **a)** Graph the function $g(x) = 5\cos(2x) + 7$ using a graphing calculator. Adjust the WINDOW settings so that $0° \leq x \leq 360°$ and $0 \leq g(x) \leq 15$. Determine the period, equation of the axis, amplitude, and range of the function.
b) Explain why the function is sinusoidal.
c) Calculate $g(125)$.
d) Determine the values of x, $0° \leq x \leq 360°$, for which $g(x) = 12$.

4. Determine the coordinates of the new point after a rotation of 64° about (0, 0) from the point (7, 0).

Lesson 6.3

5. Two white marks are made on a car tire by a parking meter inspector. One mark is made on the outer edge of the tire; the other mark is made a few centimetres from the edge. The two graphs show the relationship between the heights of the white marks above the ground in terms of time as the car moves forward.



a) What is the period of each function, and what does it represent in this situation?
b) What is the equation of the axis of each function, and what does it represent in this situation?
c) What is the amplitude of each function, and what does it represent in this situation?
d) Determine the range of each function.
e) Determine the speed of each mark, in centimetres per second.
f) If a third mark were placed on the tire but closer to the centre, how would the graph of this function compare with the other two graphs?

6. The position, $P(d)$, of the Sun at sunset, in degrees north or south of due west, depends on the latitude and the day of the year, d. For a specific latitude, the position in terms of the day of the year can be modelled by the function
$P(d) = 28\sin\left(\frac{360}{365}d - 81\right)^{\circ}$.
a) Graph the function using a graphing calculator and adjust the WINDOW settings as required.
b) What is the period of the function, and what does it represent in this situation?
c) What is the equation of the axis of the function, and what does it represent in this situation?
d) What is the amplitude of the function, and what does it represent in this situation?
e) Determine the range of the function.
f) What is the angle of sunset on February 15?

6.4 Exploring Transformations of Sinusoidal Functions

YOU WILL NEED

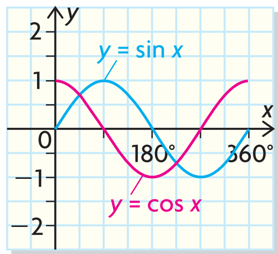
- graphing calculator

GOAL

Determine how changing the values of a, c, d, and k affect the graphs of $f(x) = a\sin(k(x - d)) + c$ and $f(x) = a\cos(k(x - d)) + c$

EXPLORE the Math

Paula and Marcus know how various transformations affect several types of functions, such as $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$.



They want to know if these same transformations can be applied to $y = \sin x$ and $y = \cos x$, and if so, how the equations and graphs of these functions change.

? Can transformations be applied to sinusoidal functions in the same manner, and do they have the same effect on the graph and the equation?

Part 1 The graphs of $y = a\sin x$ and $y = a\cos x$

A. Predict what the graphs of $y = a\sin x$, $0° \le x \le 720°$, will look like for $a = 1$, 2, and 3 and for $a = \frac{1}{2}$ and $a = \frac{1}{4}$. Sketch the graphs on the same axes. Verify your sketches using a graphing calculator.

B. On a new set of axes, repeat part A for the graphs of $y = a\sin x$, $0° \le x \le 720°$, for $a = -1$, -2, and -3.

C. How do the graphs in part A compare with those in part B? Discuss how the zeros, amplitude, and maximum or minimum values change for each function.

D. Repeat parts A to C using $y = a\cos x$.

E. Explain how the value of a affects the graphs of $y = a\sin x$ and $y = a\cos x$.

Tech Support

For Parts 1 and 2, verify your sketches by graphing the parent function ($y = \sin x$ or $y = \cos x$) in Y1 and each transformed function in Y2, Y3, and so on. Use an Xscl = 90°, and graph using ZoomFit by pressing ZOOM 0.

Part 2 The graphs of $y = \sin x + c$ and $y = \cos x + c$

F. Predict what the graphs of $y = \sin x + c$, $0° \le x \le 720°$, will look like for $c = -2$, -1, 1, and 2. Sketch the graphs on the same axes, and then verify your predictions using a graphing calculator. Discuss which features of the graph have changed.

G. Predict what the graphs of $y = \cos x + c$, $0° \le x \le 720°$, will look like for $c = -2$, -1, 1, and 2. Sketch the graphs on the same axes, and then verify your predictions using a graphing calculator. Discuss which features of the graph have changed.

H. Explain how the value of c affects the graphs of $y = a\sin x + c$ and $y = a\cos x + c$.

Tech Support

For Part 3, verify your sketches by graphing the parent function in Y1 and each transformed function in Y2. Use an Xscl = 90° and graph using ZoomFit.

Part 3 The graphs of $y = \sin kx$ and $y = \cos kx$

I. Predict what the graphs of $y = \sin kx$ will look like for $k = 2, 3$, and 4, $0° \leq x \leq 720°$. Sketch each graph, and then verify your predictions using a graphing calculator. Discuss which features of the graph have changed. Clear the previous equation, but not the base equation, from the graphing calculator before entering another equation.

J. Repeat part I for $k = \frac{1}{2}$, $k = \frac{1}{4}$, and $k = -1$. Adjust the WINDOW on the graphing calculator so that you can see one complete cycle of each graph.

K. Repeat parts I and J using $y = \cos kx$.

L. How could you determine the period of $y = \sin kx$ and $y = \cos kx$ knowing that the period of both functions is 360°?

M. Explain how the value of k affects $y = \sin kx$ and $y = \cos kx$.

i)

x	y
60°	
150°	
240°	
330°	
420°	

ii)

x	y
−120°	
−30°	
60°	
150°	
240°	

Part 4 The graphs of $y = \sin(x - d)$ and $y = \cos(x - d)$

N. **a)** Predict the effect of d on the graph of $y = \sin(x - d)$.

b) Copy and complete the tables of values at the left.

i) $y = \sin(x - 60°)$ **ii)** $y = \sin(x + 120°)$

c) Use your tables to sketch the graphs of the two sinusoidal functions from part (b) on the same coordinate system. Include the graph of $y = \sin x$. Verify your sketches using a graphing calculator, and discuss which features of the graph have changed.

i)

x	y
−45°	
45°	
135°	
225°	
315°	

ii)

x	y
120°	
210°	
300°	
390°	
480°	

O. **a)** Predict the effect of d on the graph of $y = \cos(x - d)$.

b) Copy and complete the tables of values at the left.

i) $y = \cos(x + 45°)$ **ii)** $y = \cos(x - 120°)$

c) Use your tables to sketch the graphs of the two sinusoidal functions from part (b) on the same coordinate system. Include the graph of $y = \cos x$. Verify your sketches with a graphing calculator, and discuss which features of the graph have changed.

P. Explain how the value of d affects the graphs of $y = \sin(x - d)$ and $y = \cos(x - d)$.

Tech Support

For Part 4, verify your sketches using a domain of $0° \leq x \leq 360°$ and an Xscl = 30°. Graph using ZoomFit.

Reflecting

Q. What transformation affects the period of a sinusoidal function?

R. What transformation affects the equation of the axis of a sinusoidal function?

S. What transformation affects the amplitude of a sinusoidal function?

T. What transformations affect the location of the maximum and minimum values of the sinusoidal function?

U. Summarize how the graphs of $y = a \sin(k(x - d)) + c$ and $y = a \cos(k(x - d)) + c$ compare with the graphs of $y = \sin x$ and $y = \cos x$.

In Summary

Key Ideas

- The graphs of the functions $f(x) = a\sin(k(x - d)) + c$ and $f(x) = a\cos(k(x - d)) + c$ are periodic in the same way that the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ are. The differences are only in the placement of the graph and how stretched or compressed it is.
- The values a, k, c, and d in the functions $f(x) = a\sin(k(x - d)) + c$ and $f(x) = a\cos(k(x - d)) + c$ affect the graphs of $y = \sin x$ and $y = \cos x$ in the same way that they affect the graphs of $y = f(k(x - d)) + c$, where $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$.

Need to Know

- Changing the value of c results in a vertical translation and affects the equation of the axis, the maximum and minimum values, and the range of the function but has no effect on the period, amplitude, or domain.
- Changing the value of d results in a horizontal translation and slides the graph to the left or right but has no effect on the period, amplitude, equation of the axis, domain, or range unless the situation forces a change in the domain or range.
- Changing the value of a results in a vertical stretch or compression and affects the maximum and minimum values, amplitude, and range of the function but has no effect on the period or domain. If a is negative, a reflection in the x-axis also occurs.
- Changing the value of k results in a horizontal stretch or compression and affects the period, changing it to $\frac{360°}{|k|}$, but has no effect on the amplitude, equation of the axis, maximum and minimum values, domain, and range unless the situation forces a change in the domain or range. If k is negative, a reflection in the y-axis also occurs.

FURTHER Your Understanding

1. State the transformation to the graph of either $y = \sin x$ or $y = \cos x$ that has occurred to result in each sinusoidal function.

a) $y = 3\cos x$
b) $y = \sin(x - 50°)$
c) $y = -\cos x$
d) $y = \sin(5x)$
e) $y = \cos x - 6$
f) $y = \cos(x + 20°)$

2. Each sinusoidal function below has undergone one transformation that has affected either the period, amplitude, or equation of the axis. In each case, determine which characteristic has been changed and indicate its value.

a) $y = \sin x + 2$
b) $y = 4\sin x$
c) $y = \cos(8x)$
d) $y = \sin(2x + 30°)$
e) $y = 0.25\cos x$
f) $y = \sin(0.5x)$

3. Which two of these transformations do not affect the period, amplitude, or equation of the axis of a sinusoidal function?

a) reflection in the x-axis
b) vertical stretch/vertical compression
c) vertical translation
d) horizontal stretch/horizontal compression
e) horizontal translation

6.5 Using Transformations to Sketch the Graphs of Sinusoidal Functions

YOU WILL NEED

- graph paper

GOAL

Sketch the graphs of sinusoidal functions using transformations.

LEARN ABOUT the Math

Glen has been asked to graph the sinusoidal function $f(x) = 3\sin(2(x - 60°)) + 4$ without using technology.

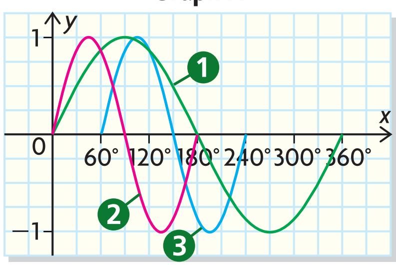
? How can you graph sinusoidal functions using transformations?

EXAMPLE 1 Using transformations to sketch the graph of a sinusoidal function

Sketch the graph of $f(x) = 3\sin(2(x - 60°)) + 4$.

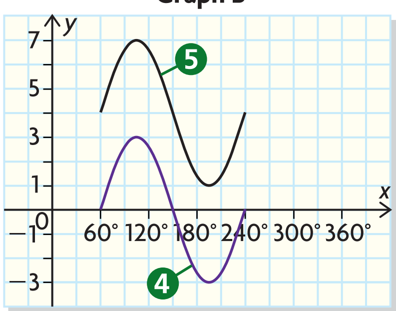
Glen's Solution

Graph A



1. I started by graphing $y = \sin x$ (in green).
2. Then I graphed $y = \sin(2x)$ (in red). It has a horizontal compression of $\frac{1}{2}$, so the period is $\frac{360°}{2} = 180°$ instead of 360° because all the x-coordinates of the points on the graph of $y = \sin x$ have been divided by 2.
3. Then I graphed $y = \sin(2(x - 60°))$ (in blue) by applying a horizontal translation of $y = \sin(2x)$ 60° to the right because 60° has been added to all the x-coordinates of the points on the previous graph.

Graph B



4. Next, I graphed $y = 3\sin(2(x - 60°))$ (in purple) by applying a vertical stretch of 3 to $y = \sin(2(x - 60°))$. The amplitude is now 3 because all the y-coordinates of the points on the previous graph have been multiplied by 3.
5. Finally, I graphed $y = 3\sin(2(x - 60°)) + 4$ (in black) by applying a vertical translation of 4 to $y = 3\sin(2(x - 60°))$. This means that the whole graph has slid up 4 units and that the equation of the axis is now $y = 4$ because 4 has been added to all the y-coordinates of the points on the previous graph.

Reflecting

A. In what order were the transformations applied to the function $y = \sin x$?

B. If the equation of the function $y = 3\sin(2(x - 60°)) + 4$ were changed to $y = 3\sin(2(x - 60°)) - 5$, how would the graph of the function change? How would it stay the same?

C. If the equation of the function $y = 3\sin(2(x - 60°)) + 4$ were changed to $y = 3\sin(9(x - 60°)) + 4$, how would the graph of the function change?

D. Which transformations affect the range of the function? How?

E. Which transformations affect the period of the function? How?

F. Could Glen graph this function faster by combining transformations? If so, which ones?

APPLY the Math

EXAMPLE 2 Connecting transformations to the graph of a sinusoidal function

a) Graph $y = -2\cos(3x) - 1$ using transformations.

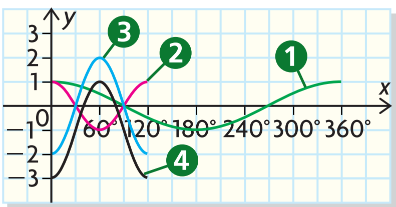
b) State the amplitude, period, equation of the axis, phase shift, and range of this sinusoidal function.

phase shift

the horizontal translation of a sinusoidal function

Steven's Solution

a)



1 I started by graphing $y = \cos x$ (in green).

2 I dealt with the horizontal compression first. I graphed $y = \cos(3x)$ (in red) using a period of $\frac{360°}{3} = 120°$ instead of 360°.

3 I dealt with the vertical stretch and the reflection in the x-axis. I graphed $y = -2\cos(3x)$ (in blue) starting at its lowest value due to the reflection, changing its amplitude to 2 due to the vertical stretch.

b) The amplitude is 2.

The period is 120°.

The equation of the axis is $y = -1$.

The phase shift is 0.

The range is $\{y \in \mathbf{R} \mid -3 \leq y \leq 1\}$.

4 Finally, I did the vertical translation. I graphed $y = -2\cos(3x) - 1$ (in black) by sliding the previous graph down 1 unit, so the equation of the axis is $y = -1$.

You can graph sinusoidal functions more efficiently if you combine and use several transformations at the same time.

EXAMPLE 3 Using a factoring strategy to determine the transformations

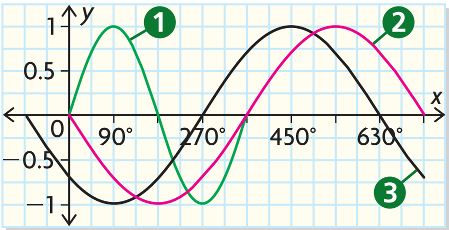
Graph $y = -\sin(0.5x + 45°)$ using transformations.

John's Solution

$y = -\sin(0.5x + 45°)$

$y = -\sin[0.5(x + 90°)]$

I factored the expression inside the brackets so that I could see all the transformations. I divided out the common factor 0.5 from 0.5*x* and 45.



1. I started by graphing $y = \sin x$ (in green).
2. Rather than graph this one transformation at a time, I dealt with all stretches/compressions and reflections at the same time. I graphed $y = -\sin(0.5x)$ (in red) by using a period of $\frac{360°}{0.5} = 720°$ and reflecting this across the *x*-axis.
3. I applied the phase shift and graphed $y = -\sin(0.5(x + 90°))$ (in black) by shifting all the points on the previous graph 90° to the left.

In Summary

Key Idea

- Functions of the form $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$, respectively, one at a time, following the order of operations (multiplication and division before addition and subtraction) for all vertical transformations and for all horizontal transformations. The horizontal and vertical transformations can be completed in either order.
- As with other functions, you can apply all stretches/compressions and reflections together followed by all translations to graph the transformed function more efficiently.

Need to Know

- To graph $g(x)$, you need to apply the transformations to the key points of $f(x) = \sin x$ or $f(x) = \cos x$ only, not to every point on $f(x)$.
  - Key points for $f(x) = \sin x$
    (0°, 0), (90°, 1), (180°, 0), (270°, −1), (360°, 0)
  - Key points for $f(x) = \cos x$
    (0°, 1), (90°, 0), (180°, −1), (270°, 0), (360°, 1)

(*continued*)

- By doing so, you end up with a function with
  - an amplitude of $|a|$
  - a period of $\frac{360°}{|k|}$
  - an equation of the axis $y = c$
- Horizontal and vertical translations of sine and cosine functions can be summarized as follows:

  Horizontal
  - Move the graph d units to the right when $d > 0$.
  - Move the graph $|d|$ units to the left when $d < 0$.

  Vertical
  - Move the graph $|c|$ units down when $c < 0$.
  - Move the graph c units up when $c > 0$.
- Horizontal and vertical stretches of sine and cosine functions can be summarized as follows:

  Horizontal
  - Compress the graph by a factor $\left|\frac{1}{k}\right|$ when $|k| > 1$.
  - Stretch the graph by a factor $\left|\frac{1}{k}\right|$ when $0 < |k| < 1$.
  - Reflect the graph in the y-axis if $k < 0$.

  Vertical
  - Stretch the graph by a factor $|a|$ when $|a| > 1$.
  - Compress the graph by a factor $|a|$ when $0 < |a| < 1$.
  - Reflect the graph in the x-axis if $a < 0$.

CHECK *Your Understanding*

1. State the transformations, in the order you would apply them, for each sinusoidal function.

a) $f(x) = \sin(4x) + 2$

b) $y = 0.25 \cos(x - 20°)$

c) $g(x) = -\sin(0.5x)$

d) $y = 12 \cos(18x) + 3$

e) $f(x) = -20 \sin\left[\frac{1}{3}(x - 40°)\right]$

2. If the function $f(x) = 4 \cos 3x + 6$ starts at $x = 0$ and completes two full cycles, determine the period, amplitude, equation of the axis, domain, and range.

3. Use transformations to predict what the graph of $g(x) = 5 \sin(2(x - 30°)) + 4$ will look like. Verify with a graphing calculator.

PRACTISING

4. State the transformations in the order you would apply them for each sinusoidal function.

a) $y = -2\sin(x + 10°)$

b) $y = \cos(5x) + 7$

c) $y = 9\cos(2(x + 6°)) - 5$

d) $g(x) = \frac{1}{5}\sin(x - 15°) + 1$

e) $h(x) = -\sin\left[\frac{1}{4}(x + 37°)\right] - 2$

f) $d = -6\cos(3t) + 22$

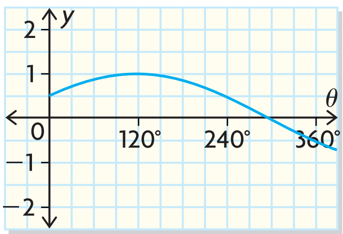
5. Match each function to its corresponding graph.

a) $y = \sin(2\theta - 90°), 0° \leq \theta \leq 360°$

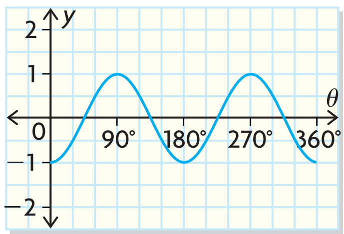
b) $y = \sin(3\theta - 90°), 0° \leq \theta \leq 360°$

c) $y = \sin\left(\frac{\theta}{2} + 30°\right), 0° \leq \theta \leq 360°$

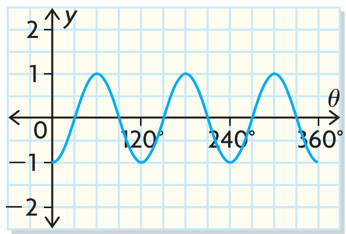
i)



ii)



iii)



6. If each function starts at $x = 0$ and finishes after three complete cycles, determine the period, amplitude, equation of the axis, domain, and range of each without graphing.

a) $y = 3\sin x + 2$

b) $g(x) = -4\cos(2x) + 7$

c) $h = -\frac{1}{2}\sin t - 5$

d) $h(x) = \cos(4(x - 12°)) - 9$

e) $d = 10\sin(180(t - 17°)) - 30$

f) $j(x) = 0.5\sin(2x - 30°)$

7. **K** Predict what the graph of each sinusoidal function will look like by describing the transformations of $y = \sin x$ or $y = \cos x$ that would result in the new graph. Sketch the graph, and then verify with a graphing calculator.

a) $y = 2\sin x + 3$

b) $y = -3\cos x + 5$

c) $y = -\sin(6x) + 4$

d) $y = 4\cos(2x) - 3$

e) $y = \frac{1}{2}\cos(3x - 120°)$

f) $y = -8\sin\left[\frac{1}{2}(x + 50°)\right] - 9$

8. Determine the appropriate WINDOW settings on your graphing calculator that enable you to see a complete cycle for each function. There is more than one acceptable answer.

a) $k(x) = -\sin(2x) + 6$

b) $j(x) = -5\sin\left(\frac{1}{2}x\right) + 20$

c) $y = 7\cos(90(x - 1°)) + 82$

d) $f(x) = \frac{1}{2}\sin(360x + 72°) - 27$

9. Each person's blood pressure is different. But there is a range of blood **A** pressure values that is considered healthy. For a person at rest, the function $P(t) = -20\cos(300t)^\circ + 100$ models the blood pressure, $P(t)$, in millimetres of mercury at time t seconds.

- **a)** What is the period of the function? What does the period represent for an individual?
- **b)** What is the range of the function? Explain the meaning of the range in terms of a person's blood pressure.

10. **T** **a)** Determine the equation of a sine function that would have the range $\{y \in \mathbf{R} \mid -1 \le y \le 7\}$ and a period of 720°.

- **b)** Determine the equation of the cosine function that results in the same graph as your function in part (a).

11. **C** Explain how you would graph the function $f(x) = -\frac{1}{2}\cos(120x) + 30$ using transformations.

Extending

12. If the functions $y = \sin x$ and $y = \cos x$ are subjected to a horizontal compression of 0.5, what transformation would map the resulting sine curve onto the resulting cosine curve?

13. The function $D(t) = 4\sin\left[\frac{360}{365}(t - 80)\right]^\circ + 12$ is a model of the number of hours, $D(t)$, of daylight on a specific day, t, at latitude 50° north.

- **a)** Explain why a trigonometric function is a reasonable model for predicting the number of hours of daylight.
- **b)** How many hours of daylight do March 21 and September 21 have? What is the significance of each of these days?
- **c)** How many hours of daylight do June 21 and December 21 have? What is the significance of each of these days?
- **d)** Explain what the number 12 represents in the model.

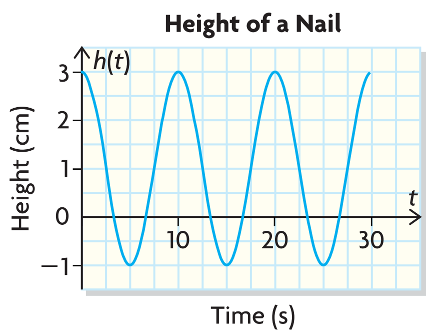
6.6 Investigating Models of Sinusoidal Functions

GOAL

Determine the equation of a sinusoidal function from a graph or a table of values.

LEARN ABOUT the Math

A nail located on the circumference of a water wheel is moving as the current pushes on the wheel. The height of the nail in terms of time can be modelled by the graph shown.



? How can you determine the equation of a sinusoidal function from its graph?

EXAMPLE 1 Representing a sinusoidal graph using the equation of a function

Determine an equation of the given graph.

Sasha's Solution

Horizontal compression factor: k

$$\text{period} = \frac{360}{|k|}$$

The period is 10 s.

$k > 0$, so $|k| = k$

I calculated the period, equation of the axis, and amplitude. Then I figured out how they are related to different transformations.

The period is 10 s since the peaks are 10 units apart. The horizontal stretch or compression factor k had to be positive because the graph was not reflected in the y-axis. I used the formula relating k to the period.

$$10 = \frac{360}{k}$$

$$k = \frac{360}{10}$$

$$k = 36$$

The graph was compressed by a factor of $\frac{1}{36}$.

Vertical translation: c

The axis is halfway between the maximum, 3, and the minimum, −1. This gave me the vertical translation and the value of c.

$$\text{equation of the axis} = \frac{\max + \min}{2}$$

$$= \frac{3 + (-1)}{2}$$

$$= 1 \text{ (vertical translation)}$$

$$c = 1$$

Vertical stretch: a

I calculated the amplitude by taking the maximum value, 3, and subtracting the axis, 1. Since the amplitude of $y = \cos x$ is 1, and the amplitude of this graph is 2, the vertical stretch is 2.

$a = 2$

Base graph: $y = \cos x$

The cosine curve is easier to use for my equation since the graph has its maximum on the y-axis, just as this graph does. This means that for a cosine curve, there isn't any horizontal translation, so $d = 0$.
I found the equation of the function by substituting the values I calculated into $f(x) = a\cos(k(x - d)) + c$.

As a cosine curve:

$y = 2\cos(36x)° + 1$

As a sine curve:

$y = 2\sin(36(x - 7.5))° + 1$

I could have used the sine function instead.

A sine curve increases from a y-value of 0 at $x = 0$.

On this graph, that happens at 7.5. This means that, for a sine curve, there is a horizontal translation of 7.5, so $c = 7.5$.

For both functions, the domain is restricted to $x \geq 0$ because it represents the time elapsed.

Reflecting

A. Tanya says that another possible equation of the sinusoidal function created by Sasha is $y = 2\cos(36(x - 10))° + 1$. Is she correct? Why or why not?

B. If the period on the original water wheel graph is changed from 10 to 20, what would be the new equation of the sinusoidal function?

C. If the maximum value on the original water wheel graph is changed from 3 to 5, what would be the new equation of the sinusoidal function?

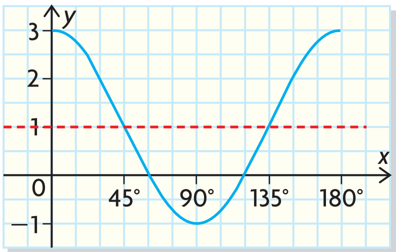
D. If the speed of the current increases so that the water wheel spins twice as fast, what would be the equation of the resulting function?

APPLY the Math

EXAMPLE 2 Connecting the equation of a sinusoidal function to its features

A sinusoidal function has an amplitude of 2 units, a period of 180°, and a maximum at (0, 3). Represent the function with an equation in two different ways.

Rajiv's Solution



The graph has a maximum at (0, 3) and a period of 180°, so the next maximum would be at (180, 3).

A minimum would be halfway between the two maximums.

Since the amplitude is 2, and $2 - 3 = -1$, the minimum would have to be at $(90°, -1)$.

Vertical translation: $c = 1$

Vertical stretch: a

The equation of the axis gave me the vertical translation. Since the equation is $y = 1$ instead of $y = 0$, there was a vertical translation of 1.

$$\text{amplitude} = 3 - 1 = 2$$
$$a = 2$$

The amplitude gives me the vertical stretch.

Horizontal compression: k

$$\text{period} = \frac{360°}{|k|}$$
$$180° = \frac{360°}{k}$$
$$k = \frac{360°}{180°}$$
$$k = 2$$

The period is 180°, so there has been a horizontal compression. Since there was no horizontal reflection, $k > 0$. To find k, I took 360° and divided it by the period.

Compression factor is $\frac{1}{2}$.

For a cosine curve:

No horizontal translation so

$d = 0$

Equation:

$y = 2\cos(2x) + 1$

Cosine curves have a maximum at $x = 0$, unless they've been translated horizontally. This curve starts at its maximum, so there would be no horizontal translation with a cosine function as a model.

I found the equation of the function by substituting the values into $f(x) = a\cos(k(x - d)) + c$.

For a sine curve:

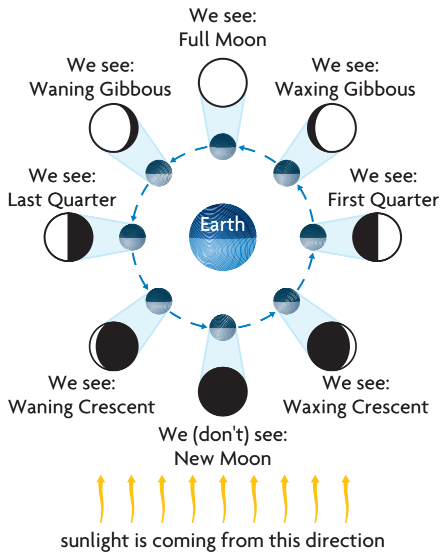
horizontal translation $= 135°$

$y = 2\sin(2(x - 135°)) + 1$

The equation of the axis of this cosine curve is $y = 1$. On this cosine curve, the point $(135°, 1)$ corresponds to the start of the cycle of the sine function. The sine curve with the same period and axis as this cosine curve has the equation $y = 2\sin(2x) + 1$, but its starting point is $(0°, 1)$. This means the function $y = 2\sin(2x) + 1$ must be translated horizontally to the right by 135°, so $d = 135°$.

EXAMPLE 3 Connecting data to the algebraic model of a sinusoidal function

The Moon is always half illuminated by the Sun. How much of the Moon we see depends on where it is in its orbit around Earth. The table shows the proportion of the Moon that was visible from Southern Ontario on days 1 to 74 in the year 2006.



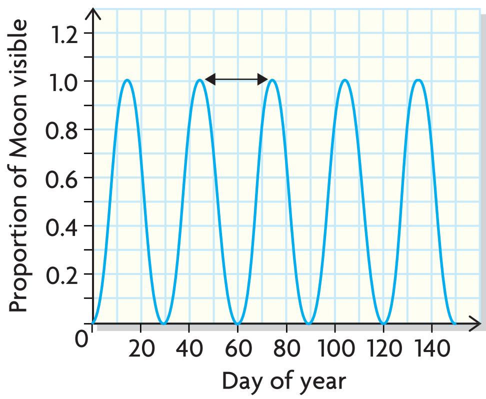
Day of Year	1	4	7	10	14	20	24	29	34
Proportion of Moon Visible	0.02	0.22	0.55	0.83	1.00	0.73	0.34	0.00	0.28

Day of Year	41	44	48	53	56	59	63	70	74
Proportion of Moon Visible	0.92	1.00	0.86	0.41	0.12	0.00	0.23	0.88	1.00

a) Determine the equation of the sinusoidal function that models the proportion of visible Moon in terms of time.
b) Determine the domain and range of the function.
c) Use the equation to determine the proportion of the Moon that is visible on day 110.

Rosalie's Solution

a) **Cycle of the Proportion of the Moon Visible**



I plotted the data. When I drew the curve, the graph looked like a sinusoidal function.

The maximum value was 1, and the minimum value 0.

The graph repeats every 30 days, so the period must be 30 days.

I figured out some of the important features of the sinusoidal function.

Vertical translation: c

Equation of the axis is $y = 0.5$.

$c = 0.5$

The axis is halfway between the maximum of 1 and the minimum of 0.

Vertical stretch: a

$$\text{amplitude} = \frac{(1 - 0)}{2}$$

The amplitude is the vertical distance between the maximum and the axis. In this case, it is 0.5, or $\frac{1}{2}$.

$$= 0.5 \quad \text{or} \quad \frac{1}{2}$$

$$a = 0.5$$

Horizontal compression: k

$$\text{period} = \frac{360}{|k|}$$

I used the period to get the compression.

$k > 0$, so $|k| = k$

$$30 = \frac{360°}{k}$$

$$k = \frac{360°}{30}$$

$$k = 12$$

Horizontal translation: d

A sine curve or a cosine curve will work. I used the cosine curve. The horizontal translation is equal to the x-coordinate of a maximum, since $y = \cos x$ has a maximum at $x = 0$. I chose the x-coordinate of the maximum closest to the origin, $x = 14$.

Using a cosine curve:

$d = 14$

$$y = \frac{1}{2}\cos(12(x - 14)°) + 0.5$$

I put the information together to get the equation.

b) domain: $\{x \in \mathbf{R} \mid 0 \leq x \leq 365\}$

range: $\{y \in \mathbf{R} \mid 0 \leq y \leq 1\}$

The domain is only the non-negative values of x up to 365, since they are days of the year. The range is 0 to 1.

c) $$y = \frac{1}{2}\cos(12(x - 14)°) + 0.5$$

Since x represents the time in days, I substituted 110 for x in the equation to calculate the amount of the Moon visible at that time. Then I solved for y.

$$= \frac{1}{2}\cos(12(110 - 14)°) + 0.5$$

$$= \frac{1}{2}\cos(1152)° + 0.5$$

$$\doteq \frac{1}{2}(0.3090) + 0.5$$

$$= 0.65$$

On day 110, 65% of the Moon is exposed.

In Summary

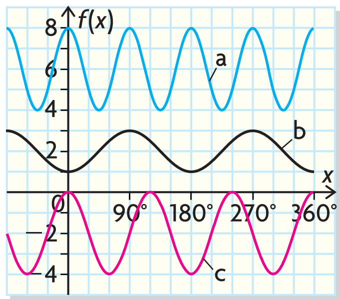
Key Idea

- If you are given a set of data and the corresponding graph is a sinusoidal function, then you can determine the equation by calculating the graph's period, amplitude, and equation of the axis. This information will help you determine the values of k, a, and c, respectively, in the equations $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$. The value of d is determined by estimating the required horizontal shift (left or right) compared with the graph of the sine or cosine curve.

Need to Know

- If the graph begins at a maximum value, it may be easier to use the cosine function as your model.
- The domain and range of a sinusoidal model may need to be restricted for the situation you are dealing with.

CHECK Your Understanding



1. Determine an equation for each sinusoidal function at the right.
2. Determine the function that models the data in the table and does not involve a horizontal translation.

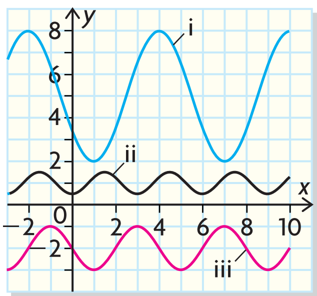
x	0°	45°	90°	135°	180°	225°	270°
y	9	7	5	7	9	7	5

3. A sinusoidal function has an amplitude of 4 units, a period of 120°, and a maximum at (0, 9). Determine the equation of the function.

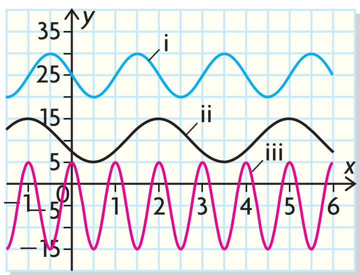
PRACTISING

4. Determine the equation for each sinusoidal function.

a)



b)



5. For each table of data, determine the equation of the function that is the simplest model.

a)

x	0°	30°	60°	90°	120°	150°	180°
y	3	2	1	2	3	2	1

b)

x	−180°	0°	180°	360°	540°	720°	900°
y	17	13	17	21	17	13	17

c)

x	−120°	−60°	0°	60°	120°	180°	240°
y	−4	−7	−4	−1	−4	−7	−4

d)

x	−20°	10°	40°	70°	100°	130°	160°
y	2	5	2	−1	2	5	2

6. Determine the equation of the cosine function whose graph has each of the
K following features.

	Amplitude	Period	Equation of the Axis	Horizontal Translation
a)	3	360°	$y = 11$	0°
b)	4	180°	$y = 15$	30°
c)	2	40°	$y = 0$	7°
d)	0.5	720°	$y = -3$	−56°

7. A sinusoidal function has an amplitude of 6 units, a period of 45°, and a minimum at (0, 1). Determine an equation of the function.

8. The table shows the average monthly high temperature for one year in
A Kapuskasing, Ontario.

Time (months)	J	F	M	A	M	J	J	A	S	O	N	D
Temperature (°C)	−18.6	−16.3	−9.1	0.4	8.5	13.8	17.0	15.4	10.3	4.4	−4.3	−14.8

a) Draw a scatter plot of the data and the curve of best fit. Let January be month 0.

b) What type of model describes the graph? Why did you select that model?

c) Write an equation for your model. Describe the constants and the variables in the context of this problem.

d) What is the average monthly temperature for month 20?

9. The table shows the velocity of air of Nicole's breathing while she is at rest.

Time (s)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2	2.25	2.5	2.75	3
Velocity (L/s)	0	0.22	0.45	0.61	0.75	0.82	0.85	0.83	0.74	0.61	0.43	0.23	0

a) Explain why breathing is an example of a periodic function.
b) Graph the data, and determine an equation that models the situation.
c) Using a graphing calculator, graph the data as a scatter plot. Enter your equation and graph. Comment on the closeness of fit between the scatter plot and the graph.
d) What is the velocity of Nicole's breathing at 6 s? Justify.
e) How many seconds have passed when the velocity is 0.5 L/s?

Tech | *Support*

For help creating a scatter plot using a graphic calculator, see Technical Appendix, B-11.

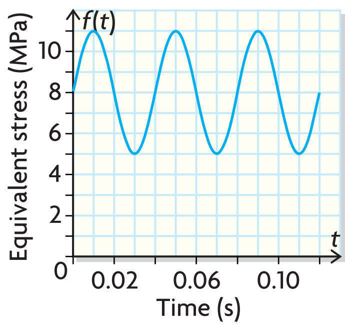
10. The table shows the average monthly temperature for three cities: Athens, Lisbon, and Moscow.

Time (month)	J	F	M	A	M	J	J	A	S	O	N	D
Athens (°C)	12	13	15	19	24	30	33	32	28	23	18	14
Lisbon (°C)	13	14	16	18	21	24	26	27	24	21	17	14
Moscow (°C)	−9	−6	0	10	19	21	23	22	16	9	1	−4

a) Graph the data to show that temperature is a function of time for each city.
b) Write the equations that model each function.
c) Explain the differences in the amplitude and the vertical translation for each city.
d) What does this tell you about the cities?

11. **C** The relationship between the stress on the shaft of an electric motor and time can be modelled with a sinusoidal function. (The units of stress are megapascals (MPa).)
a) Determine an equation of the function that describes the equivalent stress in terms of time.
b) What do the peaks of the function represent in this situation?
c) How much stress was the motor undergoing at 0.143 s?

Stress on a Motor Shaft



12. **T** Describe a procedure for writing the equation of a sinusoidal function based on a given graph.

Extending

13. The diameter of a car's tire is 60 cm. While the car is being driven, the tire picks up a nail. How high above the ground is the nail after the car has travelled 1 km?

14. Matthew is riding a Ferris wheel at a constant speed of 10 km/h. The boarding height for the wheel is 1 m, and the wheel has a radius of 7 m. What is the equation of the function that describes Matthew's height in terms of time, assuming Matthew starts at the highest point on the wheel?

6.7 Solving Problems Using Sinusoidal Models

GOAL

Solve problems related to real-world applications of sinusoidal functions.

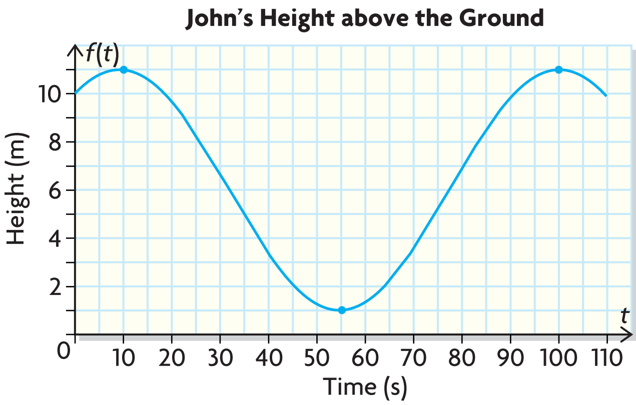
LEARN ABOUT the Math

A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches the maximum height of 11 m at 10 s and then reaches the minimum height of 1 m at 55 s.

? How can you develop the equation of a sinusoidal function that models John's height above the ground to determine his height at 78 s?

EXAMPLE 1 Connecting the equation of a sinusoidal function to the situation

Justine's Solution



I plotted the two points I knew: (10, 11) and (55, 1).

Since it takes John 45 s to go from the highest point to the lowest, then it would take him 90 s to do one complete revolution and be back to a height of 11 m at 100 s.

I drew a smooth curve to connect the points to look like a wave.

Vertical translation: c

equation of the axis:

$$y = \frac{11 + 1}{2} = 6$$

$$c = 6$$

I found the equation of the axis by adding the maximum and minimum and dividing by 2. That gave me the vertical translation and the value of c.

Vertical stretch: a

amplitude $= 11 - 6 = 5$

$$a = 5$$

I found the amplitude by taking the maximum and subtracting the y-value for the equation of the axis. That gave me the vertical stretch and the value of a.

Horizontal compression: k

For the horizontal compression, I used the formula relating the period to k. The curve wasn't reflected, so k is positive.

$$\text{period} = \frac{360}{|k|}$$

$$k > 0, \text{ so the period} = \frac{360}{k}$$

$$90 = \frac{360}{k}$$

$$k = \frac{360}{90}$$

$$k = 4$$

Horizontal translation: d

If I use the cosine function, the first maximum is at $x = 0$. The first maximum of the new function is at $x = 10$. So there was a horizontal translation of 10. That gave me the value of d.

$d = 10$

$$y = 5\cos(4(x - 10)^\circ) + 6$$

I got the equation of the sinusoidal function by substituting the values I found into $y = a\cos(k(x - d)) + c$.

$$y = 5\cos(4(78 - 10)^\circ) + 6$$
$$= 5\cos 272^\circ + 6$$
$$y \doteq 5(0.035) + 6$$
$$\doteq 6.17 \text{ m}$$

Once I had the equation, I substituted $x = 78$, and solved for the height.

At 78 s, his height will be about 6.17 m.

The answer 6.17 m looks reasonable based on the graph.

Reflecting

A. If it took John 60 s instead of 90 s to complete one revolution, how would the sinusoidal function change? State the value and type of transformation associated with this change.

B. If the radius of the Ferris wheel remained the same but the axle of the wheel was 1 m higher, how would the sinusoidal function change? State the value and type of transformation associated with this change.

C. If both characteristics from parts A and B were changed, what would be the equation of the sinusoidal function describing John's height above the ground in terms of time?

APPLY the Math

EXAMPLE 2 Solving a problem involving a sinusoidal function

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right $(+10 \text{ cm})$ and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At $t = 0$, the pole was momentarily at its resting position. Then it started moving to the right.

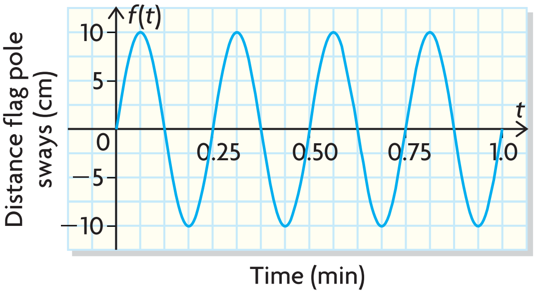
a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.

b) How does the situation affect the domain and range?

c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

Ryan's Solution

a)



I drew a graph where time is the independent variable, and the distance the top of the pole moves is the dependent variable.

The highest point on my graph will be 10, and the lowest will be −10.

I started at (0, 0) because the pole was at its resting position at $t = 0$.

$$\text{Number of sways each second} = \frac{240}{60} = 4$$

Since the pole sways back and forth 240 times in 60 s, the time to complete one sway must be 0.25 s. This is the period.

$$\text{period} = \frac{1}{4} = 0.25$$

Vertical translation: c

equation of the axis: $y = 0$

so $c = 0$

The axis is at $y = 0$. This gives the vertical translation.

Vertical stretch: a

amplitude $= 10$

so $a = 10$

I took the distance between a peak and the equation of the axis to get the amplitude.

Horizontal compression: k

I found the horizontal compression from the formula relating the period to the value of k.

$$\text{period} = \frac{360}{|k|}$$

$k > 0$

$$\text{period} = \frac{360}{|k|}$$

$$0.25 = \frac{360}{k}$$

$$k = \frac{360}{0.25}$$

$$k = 1440$$

The sine function:

I decided to use the sine function since this graph starts at $(0°, 0)$. Using the values of a and k, i determined the equation

$$y = 10 \sin(1440x)°$$

Horizontal translation: d

For the cosine function, the horizontal translation is equal to the x-coordinate of any maximum, since the maximum of a cosine function is at 0. I used the x-coordinate of the first maximum of the new function. That maximum is at $t = \frac{1}{16}$.

$$d = \frac{1}{16}$$

$$y = 10 \cos\left(1440\left(x - \frac{1}{16}\right)^{\circ}\right)$$

I put all these transformations together to get the equation of the function.

b) For either function, the domain is restricted to positive values because the values represent the time elapsed. The range of each function depends on its amplitudes.

c) 80% of 10

If the sway is the only thing that's changing, then the amplitude is going to change on the graph.
If the sway is reduced by 20%, it's 80% of what it used to be. The amplitude will then change from 10 to 8.
The vertical stretch is 8.

$$= 0.80 \times 10$$

$$= 8$$

$$y = 8 \cos\left(1440\left(x - \frac{1}{16}\right)^{\circ}\right)$$

or $y = 8 \sin(1440x)°$

In Summary

Key Idea

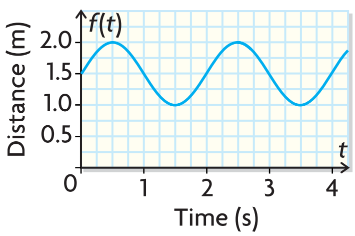
- Algebraic and graphical models of the sine and cosine functions can be used to solve a variety of real-world problems involving periodic behaviour.

Need to Know

- When you have a description of an event that can be modelled by a sinusoidal graph rather than data, it is useful to organize the information presented by drawing a rough sketch of the graph.
- You will have to determine the equation of the sinusoidal function by first calculating the period, amplitude, and equation of the axis. This information will help you determine the values of k, a, and c, respectively, in the equations $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$.

CHECK Your Understanding

Distance Between the Tail Light and the Curb



1. The load on a trailer has shifted, causing the rear end of the trailer to swing left and right. The distance from one of the tail lights on the trailer to the curb varies sinusoidally with time. The graph models this behaviour.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

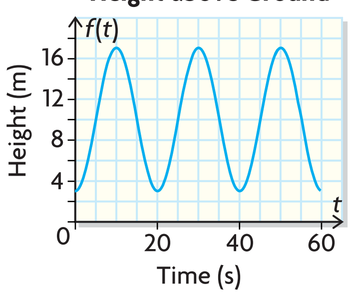
c) What is the period of the function, and what does it represent in this situation?

d) Determine the equation and the range of the sinusoidal function.

e) What are the domain and range of the function in terms of the situation?

f) How far is the tail light from the curve at $t = 3.2$ s?

Height above Ground



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

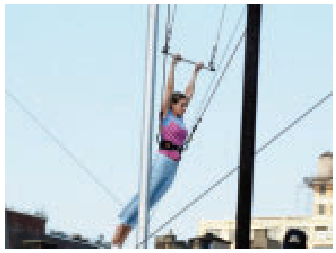
c) What is the period of the function, and what does it represent in this situation?

d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.

e) Determine the equation of the sinusoidal function.

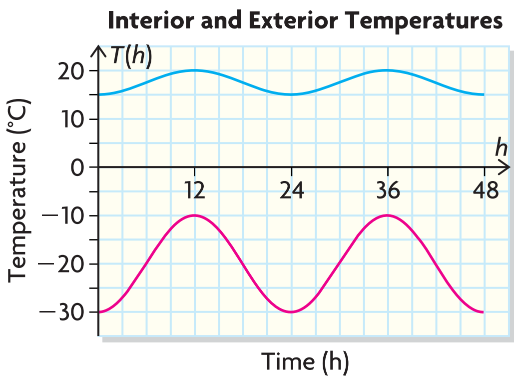
f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?

3. Chantelle is swinging back and forth on a trapeze. Her distance from a vertical support beam in terms of time can be modelled by a sinusoidal function. At 1 s, she is the maximum distance from the beam, 12 m. At 3 s, she is the minimum distance from the beam, 4 m. Determine an equation of a sinusoidal function that describes Chantelle's distance from the vertical beam in relation to time.



PRACTISING

4. **K** The interior and exterior temperatures of an igloo were recorded over a 48 h period. The data were collected and plotted, and two curves were drawn through the appropriate points.



a) How are these curves similar? Explain how each of them might be related to this situation.
b) Describe the domain and range of each curve.
c) Assuming that the curves can be represented by sinusoidal functions, determine the equation of each function.

5. **A** Skyscrapers sway in high-wind conditions. In one case, at $t = 2$ s, the top floor of a building swayed 30 cm to the left (-30 cm), and at $t = 12$, the top floor swayed 30 cm to the right $(+30 \text{ cm})$ of its starting position.
a) What is the equation of a sinusoidal function that describes the motion of the building in terms of time?
b) Dampers, in the forms of large tanks of water, are often added to the top floors of skyscrapers to reduce the severity of the sways. If a damper is added to this building, it will reduce the sway (not the period) by 70%. What is the equation of the new function that describes the motion of the building in terms of time?

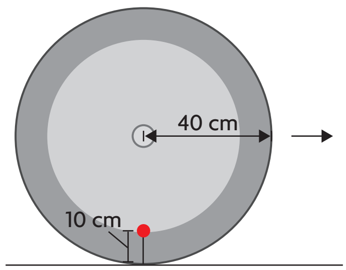
6. Milton is floating in an inner tube in a wave pool. He is 1.5 m from the bottom of the pool when he is at the trough of a wave. A stopwatch starts timing at this point. In 1.25 s, he is on the crest of the wave, 2.1 m from the bottom of the pool.
a) Determine the equation of the function that expresses Milton's distance from the bottom of the pool in terms of time.

b) What is the amplitude of the function, and what does it represent in this situation?
c) How far above the bottom of the pool is Milton at $t = 4$ s?
d) If data are collected for only 40 s, how many complete cycles of the sinusoidal function will there be?
e) If the period of the function changes to 3 s, what is the equation of this new function?

7. An oscilloscope hooked up to an alternating current (AC) circuit shows a sine curve. The device records the current in amperes (A) on the vertical axis and the time in seconds on the horizontal axis. At $t = 0$ s, the current reads its first maximum value of 4.5 A. At $t = \frac{1}{120}$ s, the current reads its first minimum value of -4.5 A. Determine the equation of the function that expresses the current in terms of time.

8. Candice is holding onto the end of a spring that is attached to a lead ball. As she moves her hand slightly up and down, the ball moves up and down. With a little concentration, she can repeatedly get the ball to reach a maximum height of 20 cm and a minimum height of 4 cm from the top of a surface. The first maximum height occurs at 0.2 s, and the first minimum height occurs at 0.6 s.
a) Determine the equation of the sinusoidal function that represents the height of the lead ball in terms of time.
b) Determine the domain and range of the function.
c) What is the equation of the axis, and what does it represent in this situation?
d) What is the height of the lead ball at 1.3 s?

9. A paintball is shot at a wheel of radius 40 cm. The paintball leaves a circular mark 10 cm from the outer edge of the wheel. As the wheel rolls, the mark moves in a circular motion.



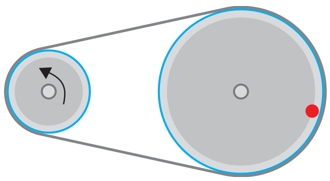
a) Assuming that the paintball mark starts at its lowest point, determine the equation of the sinusoidal function that describes the height of the mark in terms of the distance the wheel travels.
b) If the wheel completes five revolutions before it stops, determine the domain and range of the sinusoidal function.
c) What is the height of the mark when the wheel has travelled 120 cm from its initial position?

10. The population of rabbits, $R(t)$, and the population of foxes, $F(t)$, in a given
T region are modelled by the functions $R(t) = 10\,000 + 5000 \cos(15t)°$ and $F(t) = 1000 + 500 \sin(15t)°$, where t is the time in months. Referring to each graph, explain how the number of rabbits and the number of foxes are related.

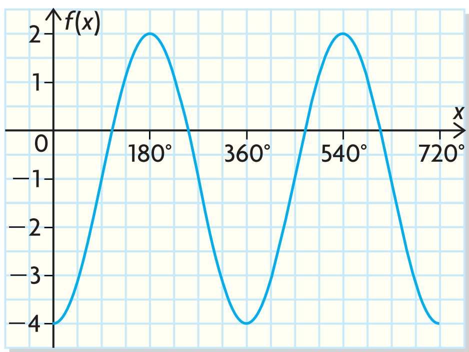
11. What information would you need to determine an algebraic or graphical
C model of a situation that could be modelled with a sinusoidal function?

Extending

12. Two pulleys are connected by a belt. Pulley A has a radius of 3 cm, and Pulley B has a radius of 6 cm. As Pulley A rotates, a drop of paint on the circumference of Pulley B rotates around the axle of Pulley B. Initially, the paint drop is 7 cm above the ground. Determine the equation of a sinusoidal function that describes the height of the drop of paint above the ground in terms of the rotation of Pulley A.



13. Examine the graph of the function $f(x)$.



a) Determine the equation of the function.
b) Evaluate $f(20)$.
c) If $f(x) = 2$, then which of the following is true for x?
  i) $180° + 360°k,\ k \in \mathbf{I}$ **iii)** $90° + 180°k,\ k \in \mathbf{I}$
  ii) $360° + 180°k,\ k \in \mathbf{I}$ **iv)** $270° + 360°k,\ k \in \mathbf{I}$
d) If $f(x) = -1$, then which of the following is true for x?
  i) $180° + 360°k,\ k \in \mathbf{I}$ **iii)** $90° + 360°k,\ k \in \mathbf{I}$
  ii) $360° + 90°k,\ k \in \mathbf{I}$ **iv)** $90° + 180°k,\ k \in \mathbf{I}$

14. Using graphing technology, determine x when $f(x) = 7$ for the function $f(x) = 4 \cos(2x) + 3$ in the domain $\{x \in \mathbf{R} \mid 0° \le x \le 360°\}$.

Curious | *Math*

Music

Pressing certain piano keys at the same time produces consonance, or pleasant sounds. Some combinations of keys produce dissonance, or unpleasant sounds.

When you strike a key, a string vibrates, causing the air to vibrate. This vibration of air produces a sound wave that your ear detects. The sound waves caused by striking various notes can be described by the functions in the table, where x is time in seconds and $f(x)$ is the displacement (or movement) of air molecules in micrometres $(1 \times 10^{-6}\text{ m})$.

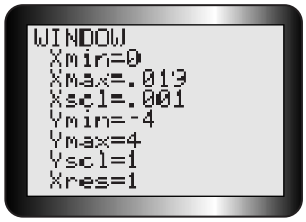
YOU WILL NEED

- graphing calculator

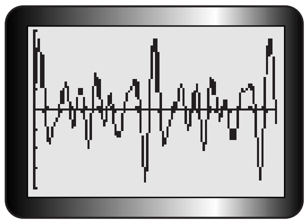
Equations for Notes (n.o. means next octave)

Note	Equation	Note	Equation	Note	Equation
A	$f(x) = \sin(158\,400x)°$	D	$f(x) = \sin(211\,427x)°$	G	$f(x) = \sin(282\,239x)°$
A#	$f(x) = \sin(167\,831x)°$	D#	$f(x) = \sin(224\,026x)°$	G#	$f(x) = \sin(299\,015x)°$
B	$f(x) = \sin(177\,806x)°$	E	$f(x) = \sin(237\,348x)°$	A n.o.	$f(x) = \sin(316\,800x)°$
C	$f(x) = \sin(188\,389x)°$	F	$f(x) = \sin(251\,465x)°$	B n.o.	$f(x) = \sin(355\,612x)°$
C#	$f(x) = \sin(199\,584x)°$	F#	$f(x) = \sin(266\,402x)°$	C n.o.	$f(x) = \sin(376\,777x)°$

One combination of notes is the A major chord, which is made up of A, C#, E, and A in the next octave. The sound can be modelled by graphing the sum of the equations for each note in Y1 using the WINDOW settings shown.



A major



1. Is the function for the A major chord periodic, sinusoidal, or both?
2. The C major chord is made up of C, E, G, and C in the next octave. Graph this function using your graphing calculator. Sketch the graph in your notebook. Compare the C major graph with the A major graph.
3. If you strike the keys A, B, C#, and F, the sound will be dissonance rather than consonance. Graph the function for this series of notes using your graphing calculator. Sketch the resulting curve. Compare with the C major and the A major graphs.
4. Graph and sketch each combination of notes below using your graphing calculator and the WINDOW settings shown above. Which combinations display consonance and which display dissonance?
   a) CC (C in first octave, C in next octave)
   b) CF
   c) CD
   d) CB (B in next octave)

6 Chapter Review

FREQUENTLY ASKED Questions

Q: How do you use transformations to determine the domain and range of a sinusoidal function?

A: The domain of a sinusoidal function is $\{x \in \mathbf{R}\}$. A restriction in the domain can occur when you consider the real-world situation you are trying to model.

To determine the range, you must determine the equation of the axis, based on the vertical translation. You then determine the amplitude, based on the vertical stretch or compression. Determine the equation of the axis, and then go above and below that value an amount equivalent to the amplitude. For example, if the equation of the axis is $y = 7$ and the amplitude is 3, then the range would be $\{y \in \mathbf{R} \mid 4 \leq y \leq 10\}$.

Study Aid

- See Lesson 6.5, Example 2.
- Try Chapter Review Question 11.

Q: How do you determine the equation of a sinusoidal function from its graph?

A: **1.** Use the formula

$$y = \frac{\text{maximum} + \text{minimum}}{2}$$

to determine the equation of the axis, which is equivalent to the vertical translation and the value of c.

2. Use the formula amplitude = maximum − axis to determine the amplitude of the function, which is equivalent to the vertical stretch or compression and the value of a. If the graph is reflected in the x-axis, then a is negative.

3. Use the formula

$$\text{period} = \frac{360°}{|k|}$$

to determine the horizontal stretch or compression, $\frac{1}{|k|}$.

4. Determine the horizontal translation. It is often easier to transform the function $y = \cos x$ than to transform $y = \sin x$ because, in many questions, it is easier to identify the coordinates of the peak of the function rather than points on the axis. If you are transforming $y = \cos x$, the horizontal translation is equivalent to the x-coordinate of any maximum. Determining this gives you the value of d.

5. Incorporate all the transformations into the equation $y = a\cos(k(x - d)) + c$ or $y = a\sin(k(x - d)) + c$.

Study Aid

- See Lesson 6.6, Example 1.
- Try Chapter Review Question 12.

PRACTICE Questions

Lesson 6.1

1. The automatic dishwasher in a school cafeteria runs constantly through lunch. The table shows the amount of water in the dishwasher at different times.

Time (min)	0	1	2	3	4	5	6	7
Volume (L)	0	16	16	16	16	16	0	16

Time (min)	8	9	10	11	12	13	14	15
Volume (L)	16	16	0	16	16	16	16	16

Time (min)	16	17	18	19	20
Volume (L)	0	16	16	16	0

a) Plot the data, and draw the resulting graph.
b) Is the graph periodic?
c) What is the period of the function, and what does it represent in this situation?
d) Determine the equation of the axis.
e) Determine the amplitude.
f) What is the range of this function?

2. Sketch a graph of a periodic function whose period is 20 and whose range is $\{y \in \mathbf{R} | 3 \leq y \leq 8\}$.

Lesson 6.2

3. Sketch the graph of a sinusoidal function that has a period of 6, an amplitude of 4, and whose equation of the axis is $y = -2$.

4. Colin is on a unique Ferris wheel: it is situated on the top of a building. Colin's height above the ground at various times is recorded in the table.

Time (s)	0	10	20	30	40	50
Height (m)	25	22.4	16	9.7	7	9.7

Time (s)	60	70	80	90	100	110
Height (m)	16	22.4	25	22.4	16	9.7

Time (s)	120	130	140	150	160
Height (m)	7	9.7	16	22.4	25

a) What is the period of the function, and what does it represent in this situation?
b) What is the equation of the axis, and what does it represent in this situation?
c) What is the amplitude of the function, and what does it represent in this situation?
d) Was the Ferris wheel already in motion when the data were recorded? Explain.
e) How fast is Colin travelling around the wheel, in metres per second?
f) What is the range of the function?
g) If the building is 6 m tall, what was Colin's boarding height in terms of the building?

Lesson 6.3

5. a) Graph the function $h(x) = 4\cos(3x) + 9$ using a graphing calculator in DEGREE mode for $0° \leq x \leq 360°$. Use Xscl $= 90°$. Determine the period, equation of the axis, amplitude, and the range of the function.
b) Is the function sinusoidal?
c) Calculate $h(45)$.
d) Determine the values of x, $0° \leq x \leq 360°$, for which $h(x) = 5$.

6. A ship is docked in port and rises and falls with the waves. The function $d(t) = 2\sin(30t)° + 5$ models the depth of the propeller, $d(t)$, in metres at t seconds. Graph the function using a graphing calculator, and answer the following questions.
a) What is the period of the function, and what does it represent in this situation?
b) If there were no waves, what would be the depth of the propeller?
c) What is the depth of the propeller at $t = 5.5$ s?
d) What is the range of the function?
e) Within the first 10 s, at what times is the propeller at a depth of 3 m?

7. Determine the coordinates of the image point after a rotation of 25° about (0, 0) from the point (4, 0).

Lesson 6.4

8. Each sinusoidal function has undergone one transformation that may have affected the period, amplitude, or equation of the axis of the function. In each case, determine which characteristic has been changed. If one has, indicate its new value.

a) $y = \sin x - 3$
b) $y = \sin(4x)$
c) $y = 7 \cos x$
d) $y = \cos(x - 70°)$

Lesson 6.5

9. Use transformations to graph each function for $0° \leq x \leq 360°$.
a) $y = 5 \cos(2x) + 7$
b) $y = -0.5 \sin(x - 30°) - 4$

10. Determine the range of each sinusoidal function without graphing.
a) $y = -3 \sin(4x) + 2$
b) $y = 0.5 \cos(3(x - 40°))$

Lesson 6.6

11. The average daily maximum temperature in Kenora, Ontario, is shown for each month.

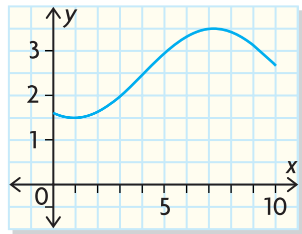
Time (months)	J	F	M	A
Temperature (°C)	−13.1	−9.0	−1.1	8.5

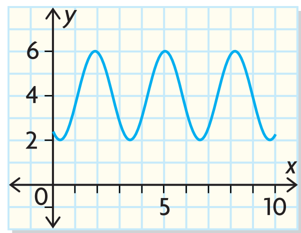
Time (months)	M	J	J	A
Temperature (°C)	16.8	21.6	24.7	22.9

Time (months)	S	O	N	D
Temperature (°C)	16.3	9.3	−1.2	−10.2

a) Prepare a scatter plot of the data. Let January represent month 0.
b) Draw a curve of good fit. Explain why this type of data can be expressed as a periodic function.
c) State the maximum and minimum values.
d) What is the period of the curve? Explain why this period is appropriate within the context of the question.
e) Write an equation for the axis of the curve.
f) What is the phase shift if the cosine function acts as the base curve?
g) Use the cosine function to write an equation that models the data.
h) Use the equation to predict the temperature for month 38. How can the table be used to confirm this prediction?

12. Determine the sine function $y = a \sin k(\theta - d) + c$ for each graph.

a)


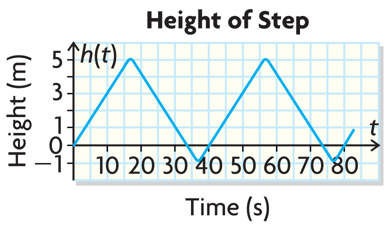
b)


Lesson 6.7

13. Meagan is sitting in a rocking chair. The distance, $d(t)$, between the wall and the rear of the chair varies sinusoidally with time t. At $t = 1$ s, the chair is closest to the wall and $d(1) = 18$ cm. At $t = 1.75$ s, the chair is farthest from the wall and $d(1.75) = 34$ cm.
a) What is the period of the function, and what does it represent in this situation?
b) How far is the chair from the wall when no one is rocking in it?
c) If Meagan rocks back and forth 40 times only, what is the domain of the function?
d) What is the range of the function in part (c)?
e) What is the amplitude of the function, and what does it represent in this situation?
f) What is the equation of the sinusoidal function?
g) What is the distance between the wall and the chair at $t = 8$ s?

14. Summarize how you can determine the equation of a sinusoidal function that represents real phenomena from data, a graph, or a description of the situation. In your summary, explain how each part of the equation relates to the characteristics of the graph.

6 Chapter Self-Test



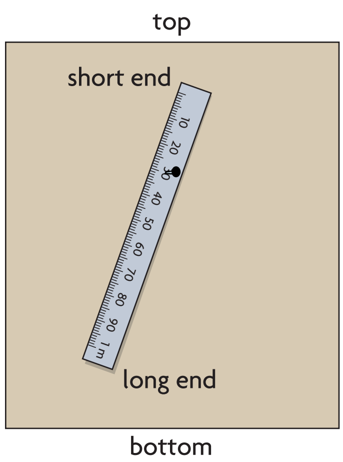
1. Steven is monitoring the height of one particular step on an escalator that takes passengers from the ground level to the second floor. The height of the step in terms of time can be modelled by the graph shown.

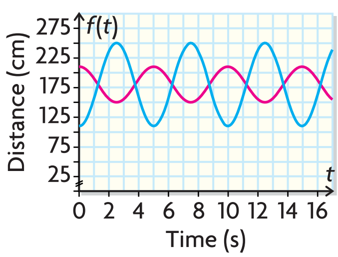
a) What is the period of the function, and what does it represent in this situation?
b) Determine the equation of the axis for this periodic function.
c) What do the peaks of the periodic function represent in this situation?
d) State the range of the function.
e) If the escalator completes only 10 cycles before being shut down, what is the domain of the periodic function?
f) Steven states that the stair will be at ground level at $t = 300$ s. Is he correct? Justify your answer.

2. Sketch a sinusoidal function that passes through $(0, -4)$ and has a period of 20, an amplitude of 3, and an equation of the axis $y = -1$.

3. Determine the coordinates of the point after a rotation of 65° about (0, 0) from the point (7, 0).

4. a) Graph $f(x) = -4\cos(0.5(x + 90°)) - 6$ using transformations of $f(x) = \cos x$.
b) State the amplitude, period, and equation of the axis.
c) Calculate $f(135°)$.
d) Determine the range of $f(x)$.





5. Keri has drilled a hole at the 30 cm mark in a metre stick. She then nails the metre stick onto a piece of plywood, through the hole. If she rotates the stick at a constant rate, then the distance from its long end to the top of the plywood can be modelled by the function in blue in the graph shown. If she rotates the stick at the same constant rate, then the distance from its short end to the top of the plywood can be modelled by the function in red.

a) What do the troughs of the sinusoidal functions represent in this situation?
b) How do the periods of the sinusoidal functions compare? Why is this so?
c) How far is the nail from the top of the plywood?
d) What is the amplitude of each sinusoidal function, and what does it represent in this situation?
e) What is the range of each sinusoidal function?
f) If Keri rotates the metre stick five complete revolutions, what is the domain of the sinusoidal function?
g) Determine the equation of each sinusoidal function.
h) What is the distance between the short end of the metre stick and the top of the plywood at $t = 19$ s?

6 Chapter Task

Cylinders and Sinusoidal Functions

? Can sinusoidal functions be obtained from cylinders?

As you follow the instructions, complete the table.

Cylinder	Radius of the Cylinder	Circumference of the Cylinder	Height of Point D	Height of Point E	Equation of the Resulting Sinusoidal Function
1					
2					
3					

A. Select one of the cylindrical objects. Determine its circumference, and record the measurement in the table. Take a sheet of paper, wrap it around the cylinder, and tape it in place. Make sure that the paper is narrow enough that the top portion of the cylinder is still exposed.

B. Mark a point D along the seam of the paper, somewhere near the top of the paper. Record the position in the table.

C. Mark a point E on the opposite side of the cylinder at least 4 cm below the height of point D. Record the position in the table.

D. Draw a line around the cylinder connecting points D and E and continue back to D.

E. Remove the cylindrically shaped object, leaving the tube of paper. Using scissors, cut along the line you drew.

F. Remove the tape and unroll the paper.

G. Determine an equation that models the resulting curve. Record the equation in the table.

H. Repeat this procedure two more times using the other cylindrical objects, marking the points D and E in different locations.

a) What is the relationship between the circumference of the cylinder and the resulting sinusoidal function?

b) What effect does changing the locations of points D and E have on the resulting sinusoidal function?

c) If you wanted to see three complete cycles on the paper, what would have to be included in the instructions?

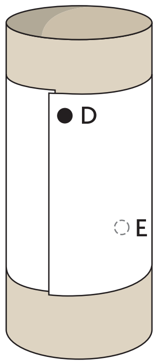
d) How could you do a similar activity and create a function that was periodic but not sinusoidal?

e) If the period of the resulting sinusoidal function was 69.12 cm, calculate the radius of the cylinder.

f) Another cylinder has a radius of 7 cm, point D at 12 cm high, and point E at 8 cm high. Determine the equation of the resulting sinusoidal function.

YOU WILL NEED

- three cylindrically shaped objects, for example, a pop can, a wooden dowel, and an empty paper towel roll
- 216 × 279 mm (letter-size) paper
- tape
- scissors



Task *Checklist*

✔ Did you show and explain the steps you used to determine the equations?

✔ Did you support your choice of data used to determine each equation?

✔ Did you explain your thinking clearly when answering the questions asked in part H?

Chapters 4–6 Cumulative Review

Multiple Choice

1. Which of the following expressions has a value of -7?

a) $25^{\frac{1}{2}} + 16^{\frac{3}{4}}$
b) $8^{\frac{2}{3}} - 81^{\frac{3}{4}} + 4^2$
c) $8^{-\frac{3}{4}} - 81^{-\frac{3}{4}} + 8^{-3}$
d) $81^{-\frac{3}{4}} + 16^{-\frac{3}{4}} - 16^{-\frac{1}{2}}$

2. Identify the expressions that are true when $x = 2$.

a) $3^{2x-1} = 27$
b) $6^{2x-3} = \sqrt{6}$
c) $5^{3x+2} = \frac{1}{5}$
d) $(2^{2x})(2^{x-1}) = 32$

3. Identify the expression that simplifies to 1.

a) $(a^{10+2p})(a^{-p-8})$
b) $(2x^2)^{3-2m}\left(\frac{1}{x}\right)^{2m}$
c) $[(c)^{2n-3m}](c^3)^m \div (c^2)^n$
d) $\left[(x^{4n-m})\left(\frac{1}{x}\right)\right]^6$

4. The population of a town is growing at an average rate of 5% per year. In 2000, its population was 15 000. What is the best estimate of the population in 2020 if the town continues to grow at this rate?

a) 40 000 **c)** 35 000
b) 30 000 **d)** 45 000

5. Point $P(-7, 24)$ is on the terminal arm of an angle in standard position. What is the measure of the related acute angle and the principal angle to the nearest degree?

a) 74° and 106° **c)** 16° and 164°
b) 16° and 344° **d)** 74° and 286°

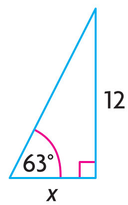
6. What is the exact value of csc 300°?

a) $\frac{\sqrt{3}}{2}$ **b)** $\frac{2}{\sqrt{3}}$ **c)** $-\frac{2\sqrt{3}}{3}$ **d)** $\frac{1}{2}$

7. Which equation is not an identity?

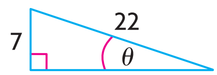
a) $(1 - \tan^2\theta)(1 - \cos^2\theta) = \dfrac{\sin^2\theta - 4\sin^4\theta}{1 - \sin^2\theta}$
b) $\dfrac{\tan x \sin x}{\tan x + \sin x} = \dfrac{\tan x \sin x}{\tan x \sin x}$
c) $\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$
d) $\dfrac{\frac{1 - \cos x}{\cos x}}{\tan x} = \dfrac{1 - \cos x}{\sin x}$

8. What is the measure of x to the nearest unit?



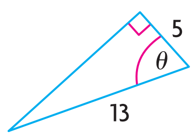
a) 4 **b)** 5 **c)** 6 **d)** 7

9. What is the measure of θ to the nearest degree?



a) 19° **b)** 22° **c)** 15° **d)** 27°

10. Which is the correct ratio for csc θ?

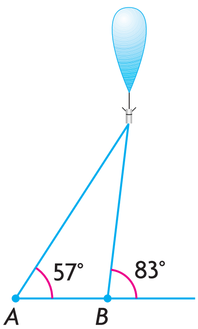


a) $\frac{5}{13}$ **b)** $\frac{13}{5}$ **c)** $\frac{13}{12}$ **d)** $\frac{12}{5}$

11. If $\tan\theta = \frac{4}{3}$ and θ lies in the third quadrant, which is the correct ratio for $\cos\theta$?

a) $\frac{4}{5}$ **b)** $-\frac{3}{5}$ **c)** $-\frac{4}{5}$ **d)** $\frac{3}{5}$

12. A weather balloon is spotted from two angles of elevation, 57° and 83°, from two different tracking stations. The tracking stations are 15 km apart. Determine the altitude of the balloon if the tracking stations and the point directly below the ballon lie along the same straight line.



a) 28.5 km
b) 32 km
c) 984 km
d) 23.7 km

13. At a concert, a spotlight is placed at a height of 12.0 m. The spotlight beam shines down at an angle of depression of 35°. How far is the spotlight from the stage?

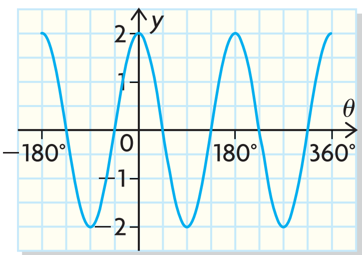
a) 20.9 m
b) 12.1 m
c) 25 m
d) 9.6 m

14. In $\triangle ABC$, $\angle A = 32°$, $\angle C = 81°$, and $a = 24.1$. Solve the triangle, and identify the correct solution.

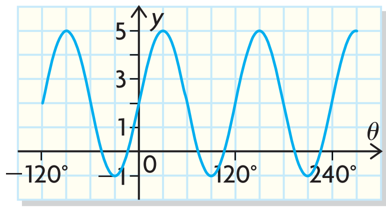
a) $\angle B = 125°$, $AC = 14.2$, $AB = 44.9$
b) $\angle B = 52°$, $AC = 41.9$, $AB = 44.9$
c) $\angle B = 107°$, $AC = 29.4$, $AB = 44.9$
d) $\angle B = 67°$, $AC = 41.9$, $AB = 44.9$

15. Which is the graph of $y = 2\cos 2(\theta + 45°) + 4$?

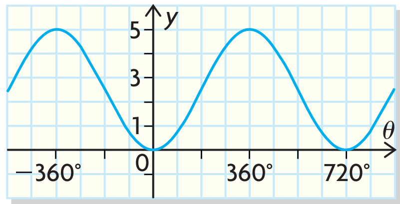
a)



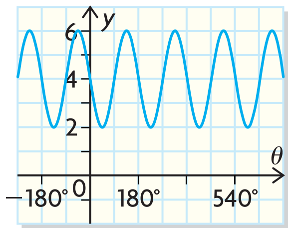
b)



c)



d)



16. Refer to the graphs in question 15. Which is the graph of $y = 2\cos 2\theta$?

a) graph a)
b) graph b)
c) graph c)
d) graph d)

17. A sine function has an amplitude of 5, a period of 720°, and range $\{y \in \mathbf{R} \mid 2 \leq y \leq 12\}$. Identify the correct equation of this function.

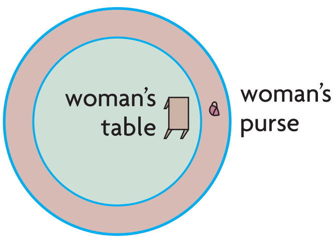
a) $y = 5\sin 2\theta + 7$
b) $y = 5\sin 2\theta - 7$
c) $y = 5\sin 0.5\theta + 7$
d) $y = 5\sin 0.5\theta - 7$

18. Identify which of the following statements is true regarding sinusoidal functions of the form $y = a\sin(k(x - d)) + c$.

a) Changing the value of a affects the maximum and minimum values, the amplitude, and the range.
b) Changing the value of k affects the amplitude, the equation of the axis, and the domain and range.
c) Changing the value of c affects the period, the amplitude, or the domain.
d) Changing the value of d affects the period, the amplitude, and the equation of the axis.

19. A circular dining room at the top of a skyscraper rotates in a counterclockwise direction so that diners can see the entire city. A woman sits next to the window ledge and places her purse on the ledge as shown. Eighteen minutes later she realizes that her table has moved, but her purse is on the ledge where she left it. The coordinates of her position are $(x, y) = (20\cos(7.5t)°, 20\sin(7.5t)°)$, where t is the time in minutes and x and y are in metres. What is the shortest distance she has to walk to retrieve her purse?

a) 54.1 m **c)** 114.0 m
b) 37.0 m **d)** 62.9 m



20. Which of the following statements is not true about the graph of $y = \sin x$?

a) The period is 360°.
b) The amplitude is 1.
c) The equation of the axis is $y = 0$.
d) The range is $\{y \in \mathbf{R} \mid 0 < y < 1\}$.

21. A regular octagon is inscribed inside a circle with a radius of 14 cm. The perimeter is

a) 32.9 cm **c)** 85.7 cm
b) 56.0 cm **d)** 42.9 cm

22. In $\triangle ABC$, $\angle A = 85°$, $c = 10$ cm, and $b = 15$ cm. A possible height of $\triangle ABC$ is

a) 10.0 cm **c)** 13.8 cm
b) 8.6 cm **d)** 12.5 cm

23. The exact value of $\cos(-420°)$ is

a) $\frac{1}{2}$ **c)** $\frac{\sqrt{3}}{2}$
b) $-\frac{\sqrt{3}}{2}$ **d)** 1

24. Using the definitions $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$, the simplified form of the expression $\dfrac{\sin^2\theta + \cos^2\theta}{\dfrac{\cos\theta}{\sin\theta}}$ is

a) $\frac{x}{y}$ **b)** $\frac{y}{x}$ **c)** $\frac{x}{r}$ **d)** $\frac{y}{r}$

25. The simplified form of the expression $\dfrac{\sin x \sin x}{(1 - \sin x)(1 + \sin x)}$ is

a) $\frac{\sin^2 x}{\cos x}$ **c)** $\tan^2 x$
b) $\frac{\sin^2 x}{\sin x}$ **d)** $\frac{\sin^2 x}{1 + \sin^2 x}$

26. The period of the function $y = \sin 4\theta$ in degrees is

a) 360° **b)** 180° **c)** 90° **d)** 1440°

27. $\left(\left(\frac{1}{a}\right)\left(\frac{1}{b^{-1}}\right)\right)^{-1}$ is equivalent to

a) $\frac{a}{b}$ **b)** $\frac{b}{a}$ **c)** $\frac{-a}{b}$ **d)** $\frac{-b}{a}$

28. If $3x^{\frac{1}{2}} = 12$, then x is equal to

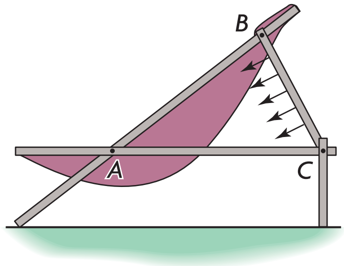
a) 576 **b)** 64 **c)** 16 **d)** $\frac{1}{64}$

Investigations

29. The Paper Folding Problem

The Paper Folding Problem was a well-known challenge to fold paper in half more than seven or eight times, using paper of any size or shape. The task was commonly known to be impossible until April 2005, when Britney Gallivan solved it.

A sheet of letter paper is about 0.1 mm thick. On the third fold it is about as thick as your fingernail. On the 7th fold it is about as thick as a notebook. If it was possible to keep folding indefinitely, how many folds would be required to end up with a thickness that surpasses the height of the CN Tower, which is 553 m?



30. Lawn Chairs

The manufacturer of a reclining lawn chair would like to have the chair positioned at the following angles: 105°, 125°, 145°, 165°, and 175°. In the figure, *AC* is 75 cm and *AB* is 55 cm. Determine the positions for the notches on *BC* that will produce the required angles. Give a complete solution.

31. Dock Dilemma

The Arps recently bought a cottage on a small, sheltered inlet on Prince Edward Island. They wish to build a dock on an outcropping of level rocks. To determine the tide's effect at this position, they measured the depth of the water every hour over a 24 h period.

Time	1:00	2:00	3:00	4:00	5:00	6:00
Depth (m)	3.81	5.05	5.94	6.25	5.89	4.95

Time	7:00	8:00	9:00	10:00	11:00	12:00
Depth (m)	3.69	2.45	1.56	1.25	1.62	2.55

Time	13:00	14:00	15:00	16:00	17:00	18:00
Depth (m)	3.81	5.05	5.94	6.25	5.89	4.95

Time	19:00	20:00	21:00	22:00	23:00	24:00
Depth (m)	3.69	2.45	1.56	1.25	1.62	2.55

a) Graph the data, and determine an equation that models this situation over a 24 h period.
b) What is the maximum depth of the water at this location?
c) The hull of their boat must have a clearance of at least 1 m at all times. Is this location suitable for their dock? Explain.